COMPUTER DESIGN OF REPEATING HYPERBOLIC PATTERNS

Douglas Dunham

University of Minnesota Duluth
Department of Computer Science
1114 Kirby Drive
Duluth, Minnesota 55812-2496 USA
ddunham@d.umn.edu

Abstract:
From antiquity, humans have created 2-dimensional art on flat surfaces (the Euclidean plane) and on surfaces of spheres. However, it wasn't until about 50 years ago that designers have created art in the third "classical geometry", the hyperbolic plane. Inspired by a diagram from the mathematician H. S. M. Coxeter, the graphic artist M. C. Escher became the first person to design such patterns, performing all the needed constructions laboriously by hand. In order to exhibit the true hyperbolic nature of such art, the pattern must exhibit symmetry and repetition. It seems natural to use a computer to avoid the tedious hand constructions performed by Escher. This was our goal: to design and implement a computer program to create repeating hyperbolic patterns.

Introduction

More than 100 years ago mathematicians created the first repeating patterns of the hyperbolic plane, triangle tessellations (see Figure 1) which were attractive, although not originally created for artistic purposes. Then in the late 1950's, the Dutch graphic artist M. C. Escher received a reprint of an article by the mathematician H. S. M. Coxeter [1]. Coxeter's paper contained a figure showing the pattern of Figure 1. Escher was inspired by this figure to create his four patterns Circle Limit I, Circle Limit II, Circle Limit III, and Circle Limit IV – see page 180 of [4]. Thus Escher became the first person to combine art and hyperbolic geometry.

It is laborious and time consuming to create repeating hyperbolic patterns by hand as Escher did. In the late 1970's, the first computer programs were written to create such patterns. Since then, considerable progress has been made in this area which spans mathematics, art, and computer science [3].

We will begin with a brief review of hyperbolic geometry and repeating patterns, followed by a discussion of regular tessellations, which form the basis for most hyperbolic patterns. Then, we will discuss symmetries, symmetry groups, color symmetry, and pattern design and replication. Finally, we will show other samples of patterns, and indicate directions of future work.

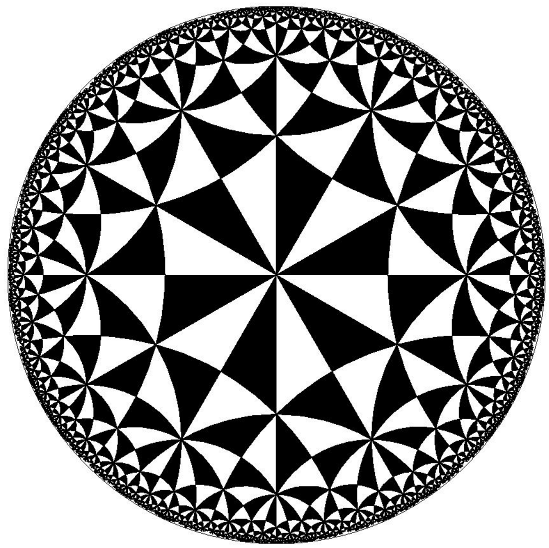

Figure 1: A pattern of triangles based on the {6,4} tessellation.

Hyperbolic Geometry and Repeating Patterns

By its definition, the geometry of the hyperbolic plane satisfies the negation of the Euclidean parallel axiom together with all the other axioms of (plane) Euclidean geometry. Consequently, hyperbolic geometry has the following parallel property: given a line *m* and a point *P* not on *m*, there is more than one line through *P* not meeting *m*. Hyperbolic geometry is not very familiar to most people, and unlike the Euclidean plane and the sphere, the entire hyperbolic plane cannot be isometrically embedded in 3-dimensional Euclidean space. Therefore, any model of hyperbolic geometry in Euclidean 3-space must distort distance.

Escher and others have used the *Poincaré circle model* of hyperbolic geometry, which has two properties that are useful for artistic purposes: (1) it is conformal (i.e. the hyperbolic measure of an angle is equal to its Euclidean measure) – consequently a transformed object has roughly the same shape as the original, and (2) it lies entirely within a circle in the Euclidean plane – allowing the viewer to see the entire hyperbolic plane. The "points" of this model are the interior points of the *bounding circle*, and the "lines" are interior circular arcs perpendicular to the bounding circle, including diameters. The sides of the triangles in Figure 1 are all hyperbolic lines, as are the edges of the hexagons in Figure 2.

However, the backbone lines in Escher's *Circle Limit III* pattern (see Figure 6 below) are *not* hyperbolic lines, but are *equidistant curves* – circular arcs making an angle of approximately 80 degrees with the bounding circle (as explained by [2], each one is a constant hyperbolic distance from the hyperbolic line with the same endpoints on the bounding circle). Because distances must be distorted in any model, equal hyperbolic distances in the Poincaré model are represented by ever smaller Euclidean distances toward the edge of the bounding circle (which is an infinite hyperbolic distance from its center). All the motifs shown in the patterns in this paper are the same hyperbolic size, even thought they are represented by different Euclidean sizes.

A *repeating pattern* of the hyperbolic plane (or the Euclidean plane or the sphere) is a pattern made up of congruent copies of a basic subpattern or *motif*. In Figure 1, any one of the triangles is a motif for that pattern – if we ignore color (that is consider the black and white triangles to be "the same"). Figure 2 shows a computer rendition in gray of Escher's first hyperbolic pattern, *Circle Limit I*, which was obviously inspired by the triangle pattern of Figure 1. Half a gray fish together with an adjacent half of a white fish form the motif for that pattern.

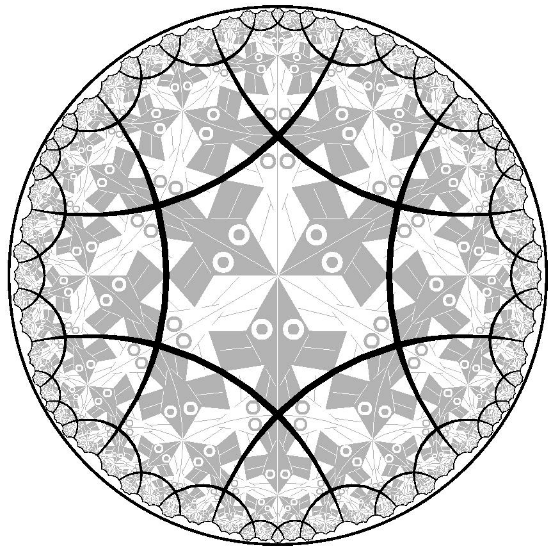

Figure 2: The {6,4} tessellation (black) superimposed on a computer-generated rendition (in gray) of Escher's *Circle Limit I* pattern.

An important kind of repeating pattern is the *regular tessellation*, denoted *{p,q}*, of the hyperbolic plane by regular *p*-sided polygons, or *p-gons*, meeting *q* at each vertex. It is necessary that $(p-2)(q-2) > 4$ for the tessellation to be hyperbolic; if $(p-2)(q-2) < 4$ one obtains a spherical tessellation, and if $(p-2)(q-2) = 4$, the tessellation is Euclidean. Figure 2 shows the {6,4} tessellation in black superimposed on a rendition of Escher's *Circle Limit I* pattern (in gray). Similarly, Figure 6 shows the {8,3} tessellation superimposed on a rendition of Escher's *Circle Limit III* pattern.

This completes our discussion of hyperbolic geometry, repeating patterns, and regular tessellations. Next, we consider the symmetry and coloring of patterns.

Symmetry Groups and Color Symmetry

Patterns with many symmetries are pleasing to the eye – and as noted before, truly hyperbolic patterns must be repeating, and so must have quite a few symmetries. A *symmetry operation* or simply a *symmetry* of a repeating pattern is an isometry (distance-preserving transformation) that transforms the pattern into itself. For example, hyperbolic reflections across the fish backbones in Figure 2 are symmetries (reflections across hyperbolic lines of the Poincaré circle model are inversions in the circular arcs representing those lines --- or ordinary Euclidean reflections across diameters). Other symmetries of Figure 2 include rotations by 180 degrees about the points where the trailing edges of fin-tips meet, and translations by four fish-lengths along backbone lines. In hyperbolic geometry, as in Euclidean geometry, a translation is the composition of successive reflections across two lines having a common perpendicular; the composition of reflections across two intersecting lines produces a rotation about the intersection point by twice the angle of intersection.

The *symmetry group* of a pattern is the set of all symmetries of the pattern. The symmetry group of the tessellation {p,q} is denoted by [p,q] and can be generated by reflections across the three sides of a right triangle with angles of $180/p$ and $180/q$ degrees; that is, all symmetries in [*p,q*] can be obtained by successively applying a finite number of those reflections. Such a right triangle is formed from a radius, an apothem, and half an edge of a *p*-gon. Those triangles corresponding to the tessellation {6,4} are shown in Figure 1, which thus has symmetry group [6,4]. The orientation-preserving subgroup of [*p,q*], consisting of symmetries composed of an even number of reflections, is denoted $[p,q]^+$. Figure 3 shows a hyperbolic pattern with symmetry group $[5,5]^+$ (ignoring color) which uses the fish motif of Escher's Notebook

Drawing Number 20 (page 131 of [5]) and his carved sphere with fish (page 244 of [5]); those patterns have symmetry groups $[4,4]^+$ and $[3,3]^+$ respectively.

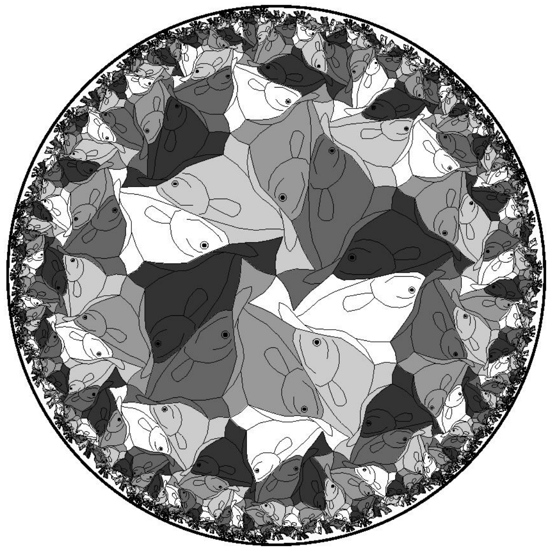

Figure 3: A hyperbolic pattern with symmetry group $[5,5]^+$ using the fish motif of Escher's Notebook Drawing Number 20.

One other subgroup of $[p,q]$, denoted $[p^+,q]$, is generated by a p-fold rotation about the center of a p-gon and a reflection in one of its sides, where q must be even so that the reflections across p-gon sides match up. Figure 4 shows a pattern of 5-armed crosses with symmetry group $[3^+,10]$ that is similar to Escher's *Circle Limit II* pattern of 4-armed crosses which has symmetry group $[3^+,8]$.

Figure 4: A hyperbolic pattern of 5-armed crosses with symmetry group $[3^+,10]$.

In these patterns, the 3-fold rotation centers are to the left and right of the ends of each cross arm, and $q/2$ reflection lines pass through the centers of the crosses (and the center of the bounding circle). Escher made similar use of the group $[p^+,q]$ for his "angel and devil" patterns in Notebook Drawing Number 45, *Heaven and Hell* on a carved maple sphere, and *Circle Limit IV*, with symmetry groups $[4^+,4]$, $[3^+,4]$, and $[4^+,6]$ respectively (see pages 150, 244, and 296 of [5]). Figure 5 shows a related pattern of devils with symmetry group $[5^+,4]$.

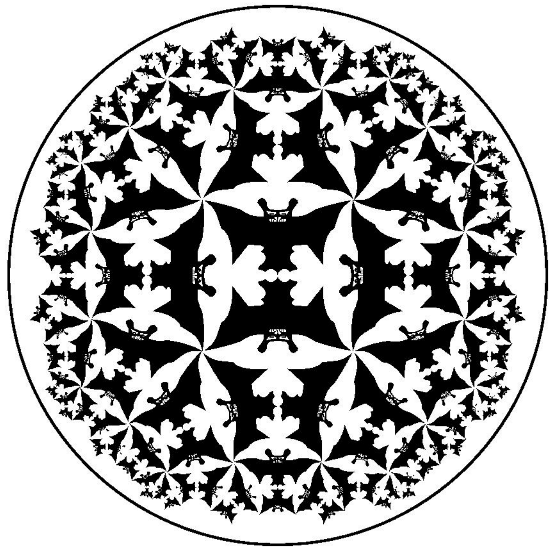

Figure 5: A hyperbolic pattern of devils with symmetry group $[5^+,4]$.

A pattern's aesthetic appeal can be enhanced by the symmetric application of color. A pattern is said to have *n-color symmetry* if each of its motifs is drawn with one of *n* colors and each symmetry of the uncolored pattern maps all motifs of one color onto motifs of another (possibly the same) color; that is, each uncolored symmetry exactly permutes the *n* colors. This concept is often called *perfect color symmetry*. Figures 1, 3, and 4 exhibit 2-, 5-, and 3-color symmetry, respectively. In contrast, note that the fish pattern of Figure 2 does not have color symmetry since the gray and white fish are not equivalent. Escher's print *Circle Limit III*, his most successful hyperbolic pattern, exhibits 4-color symmetry and is based on the tessellation {8,3}, as shown in Figure 6.

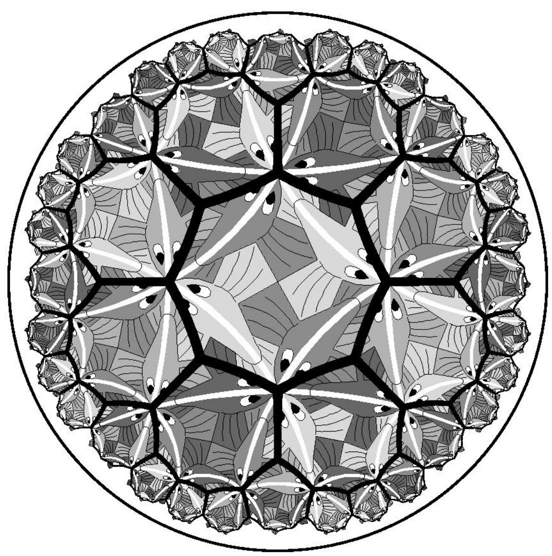

Figure 6: The {8,3} tessellation superimposed on a gray-scale rendition of Escher's *Circle Limit III* pattern.

The symmetry group of the *Circle Limit III* pattern is more complex than the examples we have seen so far --- it is generated by three rotations: a 4-fold rotation about the right fin tip, a 3-fold rotation about the left fin tip, and a 3-fold rotation about the nose of a fish. The two different kinds of 3-fold points alternate around the vertices of an octagon of the {8,3} tessellation. This completes our discussion of the theory of hyperbolic patterns. Next, we explain how these patterns can be created.

There are two phases to the pattern-creation process: the design of the motif and the replication of the entire pattern from that motif. The first step in designing a motif is to choose an underlying tessellation $\{p,q\}$, a symmetry group, and the desired color symmetry. Once these choices have been made, as an aid to the designer, the program draws the central p-gon within the bounding circle, including the radii and apothems of that p-gon. As an example, we show the design phase for the simple motif of Figure 4, which is based on the {10,3} tessellation. Figure 7 shows a motif within the central 10-gon of the {10,3} tessellation

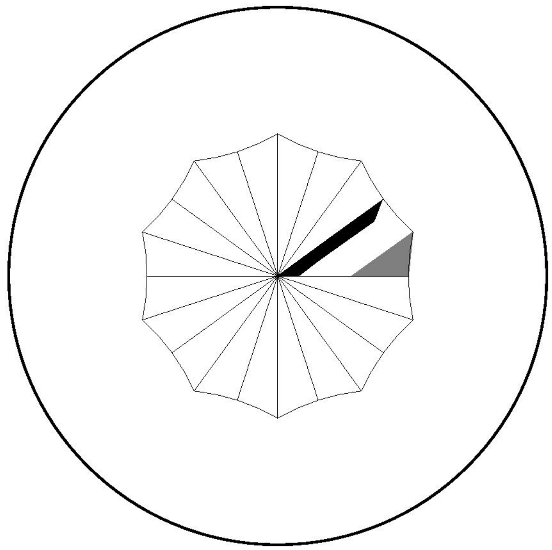

Figure 7: A motif for Figure 4 within the central 10-gon.

In order to give the motif designer an idea of how the final pattern will appear, the program shows all copies of the motif within the central p-gon. In addition, since the designer would also like to see how copies of the motif outside the central p-gon interact with the original motif, another "layer" of motifs is drawn adjacent to those in the central p-gon. Thus the designer sees all the copies of the motif (and more!) surrounding the "design motif" to the right of the center of the bounding circle – this is shown in Figure 8.

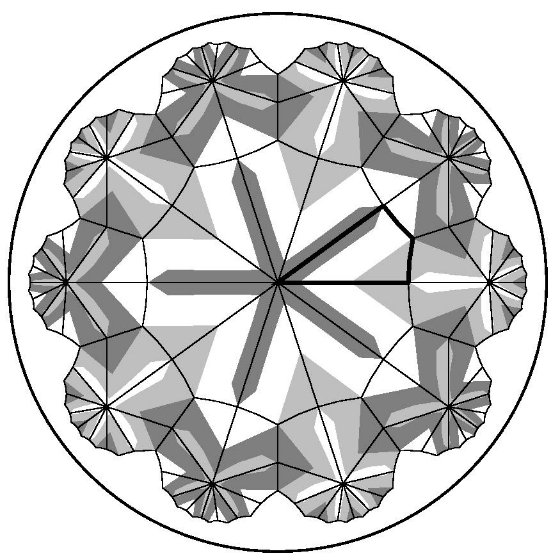

Figure 8: Copies of the motif surrounding the "design motif".

Once the motif has been designed, it can be replicated to form the final pattern. The replication phase works in two steps. First the "design motif" is copied to fill the central *p*-gon, using reflections and rotations of the symmetry group, forming what is called the *p-gon pattern*. For example, in Figure 8, the central 5-armed cross forms the *p*-gon pattern for that design.

In the second step of the replication phase, the *p*-gon pattern is transformed about the hyperbolic plane using symmetries from the symmetry group to form the final pattern. The idea behind this is to notice that the *p*-gons of a $\{p,q\}$ tessellation come in "layers". The first layer is just the central *p*-gon. The second layer consists of all those *p*-gons sharing an edge or vertex with the central *p*-gon. The third layer consists of all those *p*-gons sharing an edge or vertex with a *p*-gon in the second layer (excluding *p*-gons from the first and second layers of course). In general, the $n+1^{st}$ layer consists of all those *p*-gons sharing an edge or vertex with a *p*-gon in the *n*th layer (excluding *p*-gons from any previous layers). The superimposed tessellations in Figures 2 and 6 each contain the first three layers.

After creating the motif, the designer chooses how many layers the final pattern will have, and then the program transforms the *p*-gon pattern into each of the *p*-gons in each of the layers 1, 2, 3, etc., up to the number of layers desired. Theoretically the *p*-gon pattern should be transformed into an infinite number of layers to complete the whole pattern, but this would take an infinite amount of time. However, from a practical standpoint, this is not necessary, since the *p*-gons get close to the edge rapidly at the layer number increases, so the motifs appear to fill out the bounding circle with a small number of layers. For example, Figures 1 and 4 were drawn with just four layers. The designer can also rotate the final pattern – for example, Figure 4 is rotated 18 degrees clockwise from Figures 7 and 8. There are more details concerning the replication algorithm in [3].

Other Patterns

Our first goal in designing our program was to be able to replicate Escher's four *Circle Limit* patterns. But we succeeded in doing more than this by providing the flexibility to design any motif, to pick any *p* and *q,* to choose a symmetry group (within limitations), and to choose different color symmetries. For example, Figure 3 above was not one of Escher's *Circle Limit* patterns, nor is Figure 9, which is based on the {8,3} tessellation and Escher's Notebook Drawing Number 25.

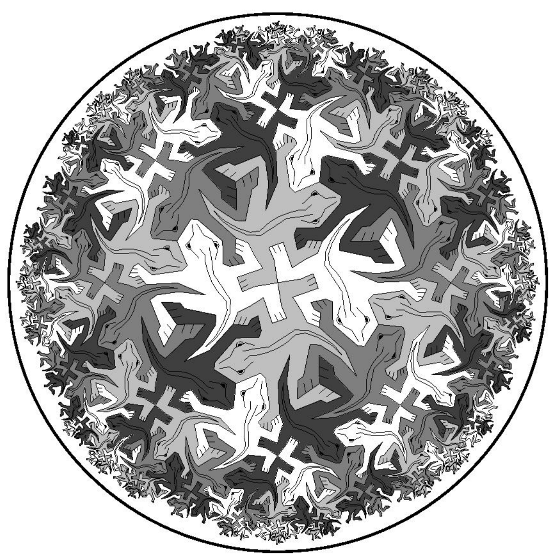

Figure 9: A hyperbolic pattern with 4-color symmetry, based on the {8,3} tessellation and the lizard motif of Escher's Notebook Drawing Number 25.

Figure 10 is another “non Circle Limit” pattern. It has 8-color symmetry, and is based on the {7,3} tessellation and the butterfly motif of Escher’s Notebook Drawing Number 70.

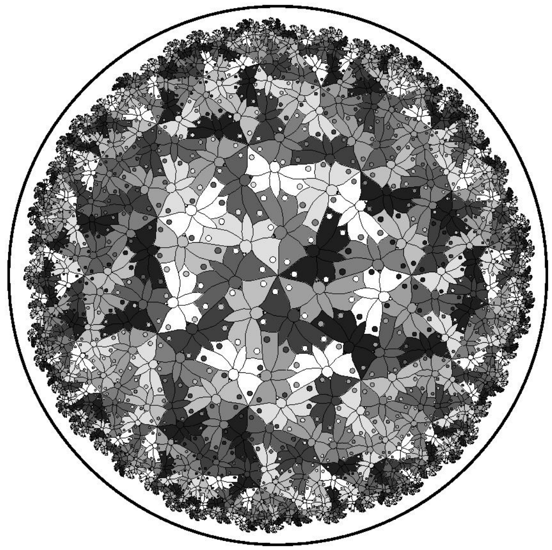

Figure 10: A hyperbolic pattern having 8-color symmetry, based on the {7,3} tessellation and the butterfly motif of Escher's Notebook Drawing Number 70.

Of course there are many other repeating Euclidean Escher patterns (he designed over 150 of them) that could be converted to hyperbolic patterns. However, this small sample finishes our consideration of other repeating hyperbolic patterns. In the final section below we indicate directions of future work.

Future Work

In addition to the many Euclidean Escher patterns remaining to be converted to hyperbolic geometry, there are infinitely many values of p and q that could be chosen for each one of those patterns, so there is much pattern-making work to be done, even if one does not venture beyond Escher’s patterns. Currently there are limitations to the kind of symmetry group that the program can use – it does not seem too difficult to remove that restriction. A more difficult task is to automate the computation of the possible color symmetries of a pattern, which must now be done by hand before designing the motif. Thus there are still many challenges remaining in the creation of artistic hyperbolic patterns.

References

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