The Use of Repeating Patterns to Teach Hyperbolic Geometry Concepts

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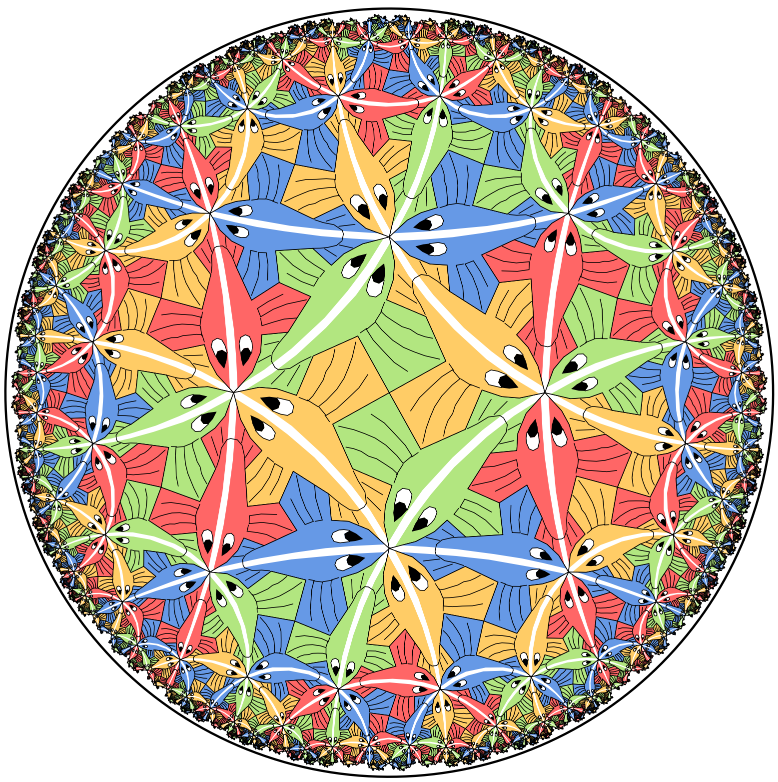


Hyperbolic Geometry — Unfamiliar

Reasons:

- **Euclidean geometry seems to describe our world — no need to negate the parallel axiom.**
- **There is no smooth, distance-preserving embedding of it in Euclidean 3-space - unlike spherical geometry.**

Outline

- **History**
- **Models and Repeating Patterns**
- **M.C. Escher's Patterns**
- **Analysis of Patterns**
- **Other Patterns**
- **Conclusions and Future Work**

The Three “Classical Geometries”

- **the Euclidean plane**
- **the sphere**
- **the hyperbolic plane**

The first two have been known since antiquity

Hyperbolic geometry has been known for slightly less than 200 years.

Hyperbolic Geometry Discovered Independently in Early 1800's by:

- **Bolyai János**
- **Carl Friedrich Gauss (did not publish results)**
- **Nikolas Ivanovich Lobachevsky**

Why the Late Discovery of Hyperbolic Geometry?

- **Euclidean geometry seemed to represent the physical world.**
- **Unlike the sphere was no smooth embedding of the hyperbolic plane in familiar Euclidean 3-space.**

Euclid and the Parallel Axiom

- **Euclid realized that the parallel axiom, the Fifth Postulate, was special.**
- **Euclid proved the first 28 Propositions in his Elements without using the parallel axiom.**

Attempts to Prove the Fifth Postulate

- **Adrien-Marie Legendre (1752–1833) made many attempts to prove the Fifth Postulate from the first four.**
- **Girolamo Saccheri (1667–1733) assumed the negation of the Fifth Postulate and deduced many results (valid in hyperbolic geometry) that he thought were absurd, and thus "proved" the Fifth Postulate was correct.**

New Geometries

- **A negation of the Fifth Postulate, allowing more than one parallel to a given line through a point, leads to another geometry, hyperbolic, as discovered by Bolyai, Gauss, and Lobachevsky.**
- **Another negation of the Fifth Postulate, allowing no parallels, and removing the Second Postulate, allowing lines to have finite length, leads to elliptic (or spherical) geometry, which was also investigated by Bolyai.**

The van Hiele Levels of Geometric Reasoning

1. **Visualization - identifying different shapes**
2. **Analysis - measurement, classification**
3. **Informal Deduction - making and testing hypotheses**
4. **Deduction - construct (Euclidean) proofs**
5. **Rigor - work with different axiom systems**

A Second Reason for Late Discovery

- **As mentioned before, there is no smooth embedding of the hyperbolic plane in familiar Euclidean 3-space (as there is for the sphere). This was proved by David Hilbert in 1901.**
- **Thus we must rely on "models" of hyperbolic geometry - Euclidean constructs that have hyperbolic interpretations.**

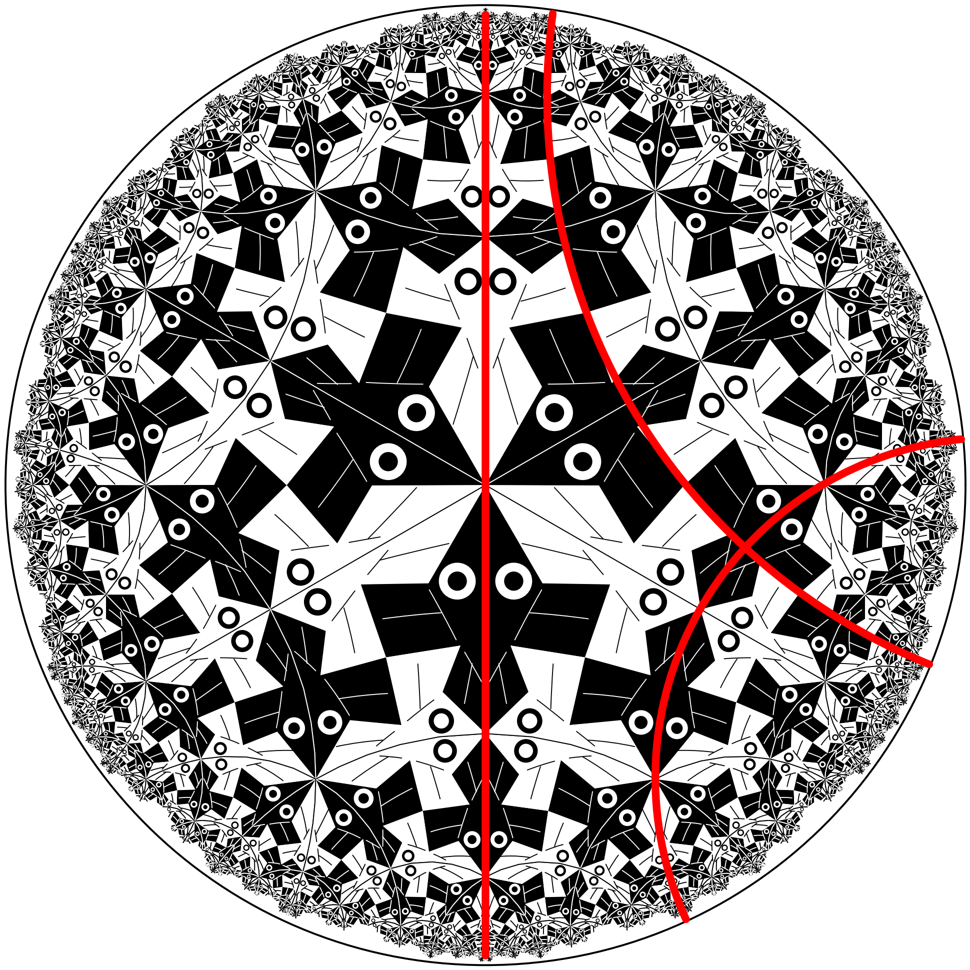
One Model: The Beltrami-Klein Model of Hyperbolic Geometry

- **Hyperbolic Points: (Euclidean) interior points of a bounding circle.**
- **Hyperbolic Lines: chords of the bounding circle (including diameters as special cases).**
- **Described by Eugenio Beltrami in 1868 and Felix Klein in 1871.**

Consistency of Hyperbolic Geometry

- **Hyperbolic Geometry is at least as consistent as Euclidean geometry, for if there were an error in hyperbolic geometry, it would show up as a Euclidean error in the Beltrami-Klein model.**
- **So then hyperbolic geometry could be the "right" geometry and Euclidean geometry could have errors!?!?**
- **No, there is a model of Euclidean geometry within 3-dimensional hyperbolic geometry — they are equally consistent.**

Poincaré Circle Model of Hyperbolic Geometry



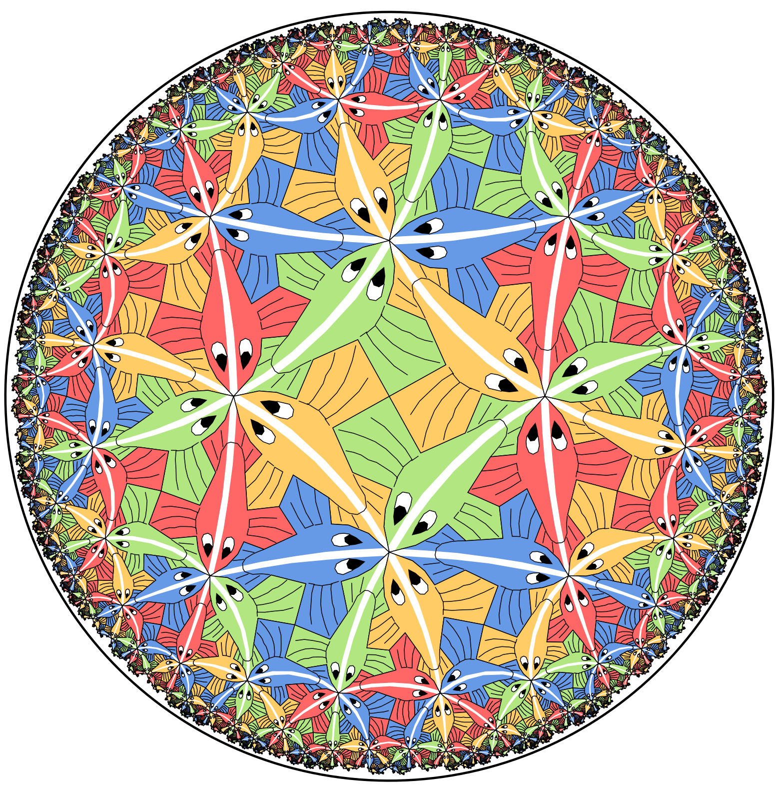
- **Points: points within the (unit) bounding circle**
- **Lines: circular arcs perpendicular to the bounding circle (including diameters as a special case)**
- **Attractive to Escher and other artists because it is represented in a finite region of the Euclidean plane, so viewers could see the entire pattern, and is *conformal*: the hyperbolic measure of an angle is the same as its Euclidean measure. This implies that copies of a motif in a repeating pattern retained their same approximate shape.**

Relation Between Poincaré and Beltrami-Klein Models

- A chord in the Beltrami-Klein model represents the same hyperbolic line as the orthogonal circular arc with the same endpoints in the Poincaré circle model.
- The models measure distance differently, but equal hyperbolic distances are represented by ever smaller Euclidean distances toward the bounding circle in both models.

Repeating Patterns

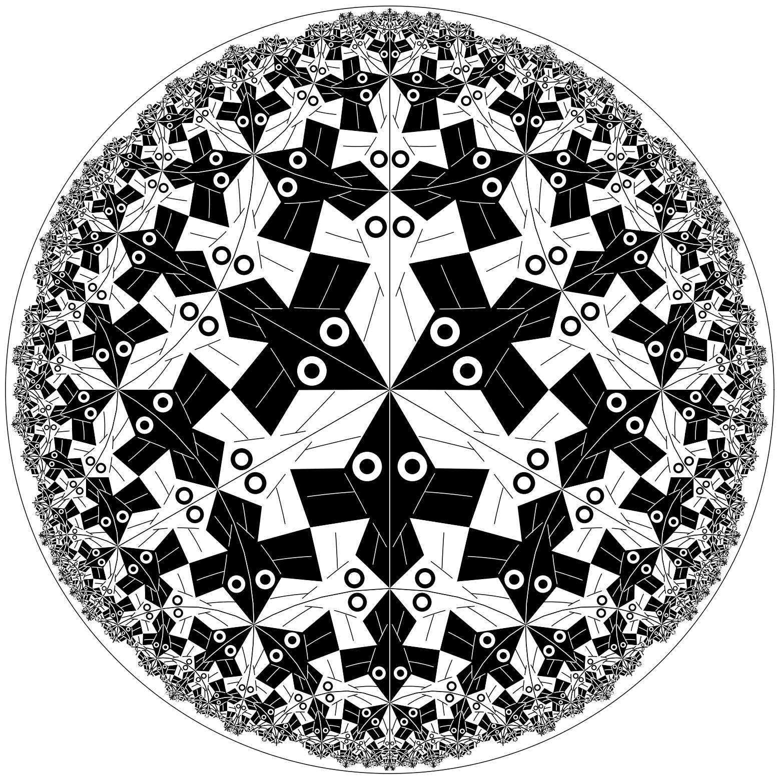
- **A *repeating pattern* in any of the three classical geometries is a pattern made up of congruent copies of a basic subpattern or *motif*.**
- **The copies of the motif are related by *symmetries* — isometries (congruences) of the pattern which map one motif copy onto another.**
- **A fish is a motif for Escher's *Circle Limit III* pattern below if color is disregarded.**



Symmetries in Hyperbolic Geometry

- A reflection is one possible symmetry of a pattern.
- In the Poincaré disk model, reflection across a hyperbolic line is represented by inversion in the orthogonal circular arc representing that hyperbolic line.
- As in Euclidean geometry, any hyperbolic isometry, and thus any hyperbolic symmetry, can be built from one, two, or three reflections.
- For example in either Euclidean or hyperbolic geometry, one can obtain a rotation by applying two successive reflections across intersecting lines, the angle of rotation being twice the angle of intersection.

Examples of Symmetries — Escher's *Circle Limit I*

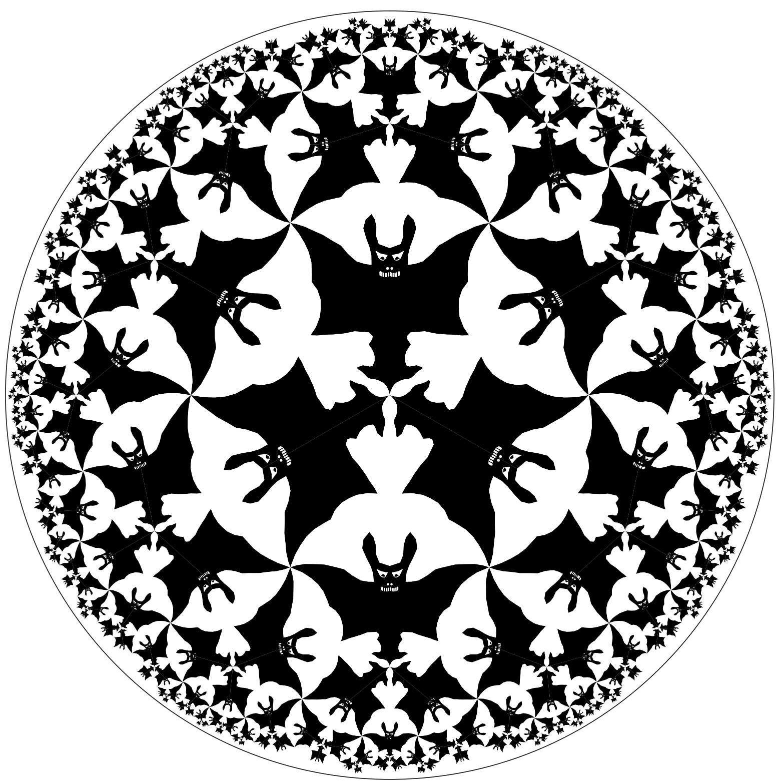


Repeating Patterns and Hyperbolic Geometry

- **Repeating patterns and their symmetry are necessary to understand the hyperbolic nature of the different models of hyperbolic geometry.**
- **For example, a circle containing a few chords may just be a Euclidean construction and not represent anything hyperbolic in the Beltrami-Klein model.**
- **On the other hand, if there is a repeating pattern within the circle whose motifs get smaller as one approaches the circle, then we may be able to interpret it as a hyperbolic pattern.**

Escher's Hyperbolic Patterns — *Circle Limit IV*

- **Only motif Escher rendered in all three classical geometries**
- **Is on display at the Portland Art Museum now.**
- **Have now seen *Circle Limit I*, *Circle Limit III*, and *Circle Limit IV* — *Circle Limit II* is below.**

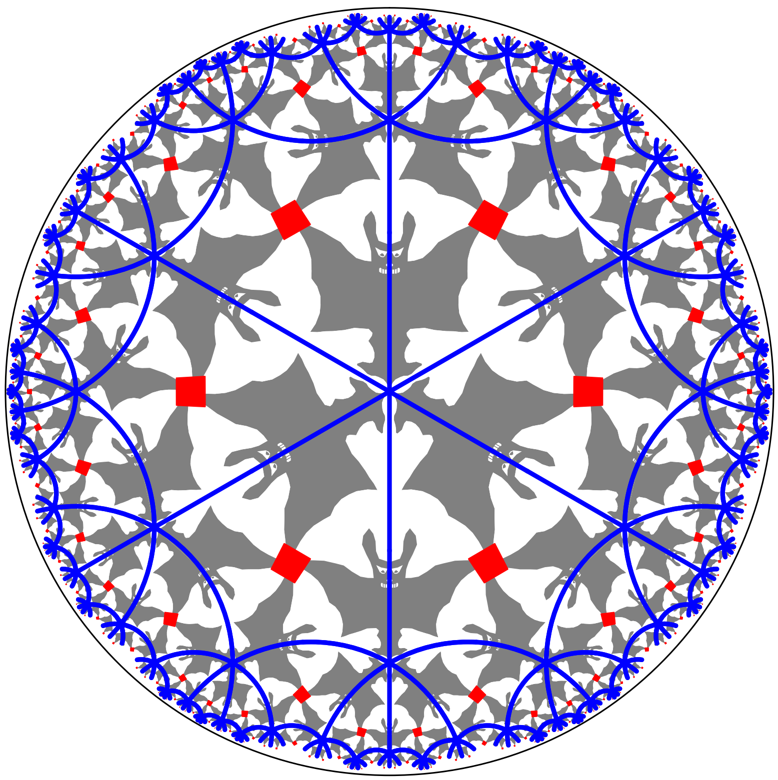


Analyzing Hyprebolic Patterns

- **There are five kinds of hyperbolic isometries:**
  - **Reflection**
  - **Rotation**
  - **Translation**
  - **Glide-Reflection**
  - **Parabolic Isometry: "rotation about a point at infinity" (does not appear in Escher's patterns)**
- **The first three are easiest to spot.**
- **As in Euclidean geometry, glide-reflections are sometimes hard to spot.**

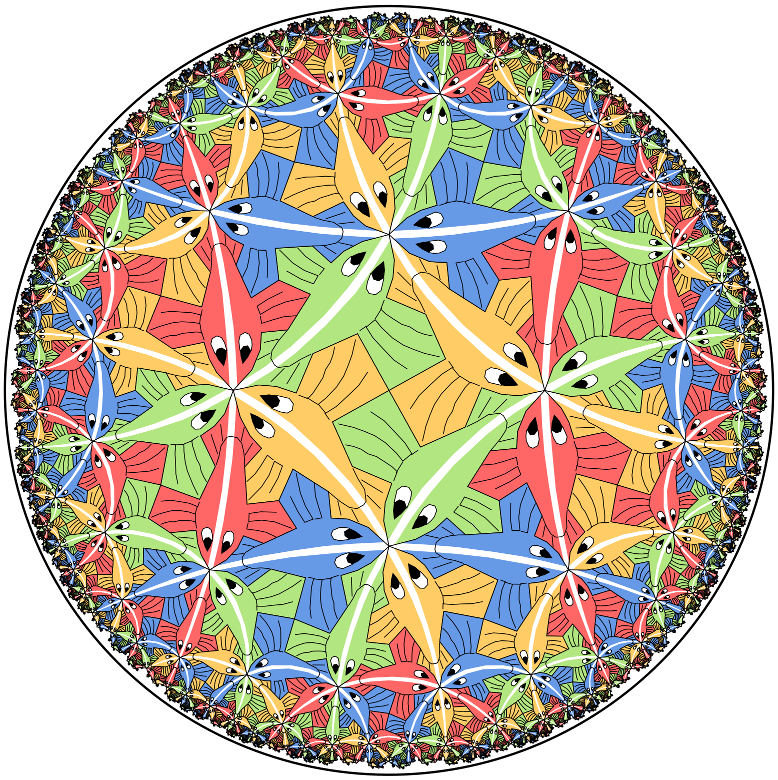
Some Symmetries of *Circle Limit IV*

- **Reflection lines shown in blue**
- 90° **rotation centers shown in red**



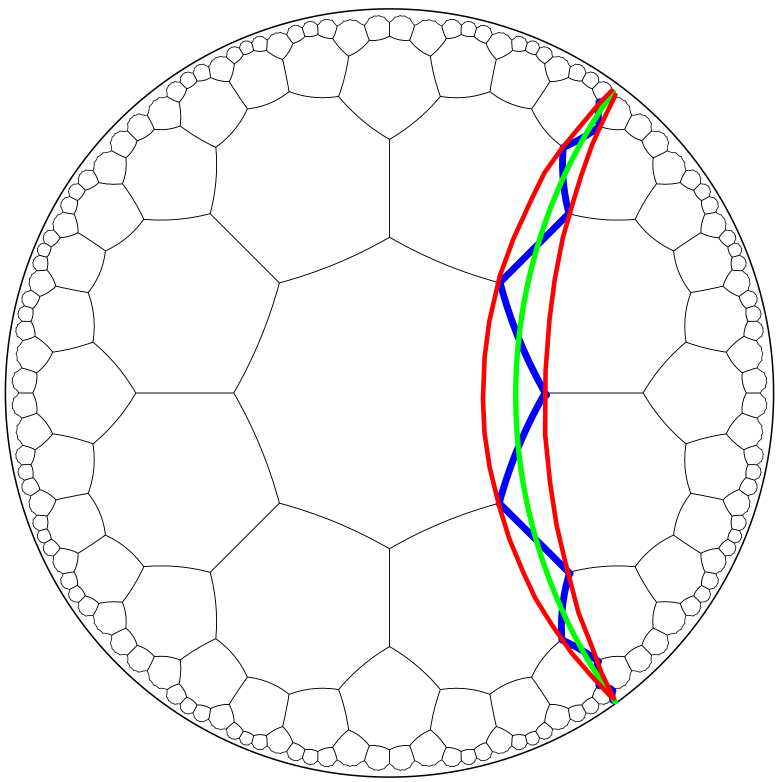
Escher's *Circle Limit III* Revisited

- **Backbone lines are *equidistant curves***
- **Analogous to lines of latitude in spherical geometry**



Equidistant Curves in *Circle Limit III*

- **Meeting points of left fins and noses are vertices of a tessellation by regular octagons meeting three at a vertex.**
- **Red backbone lines are equidistant from the green hyperbolic line that goes through the centers of the blue zigzag formed by sides of octagons.**



Color Symmetry

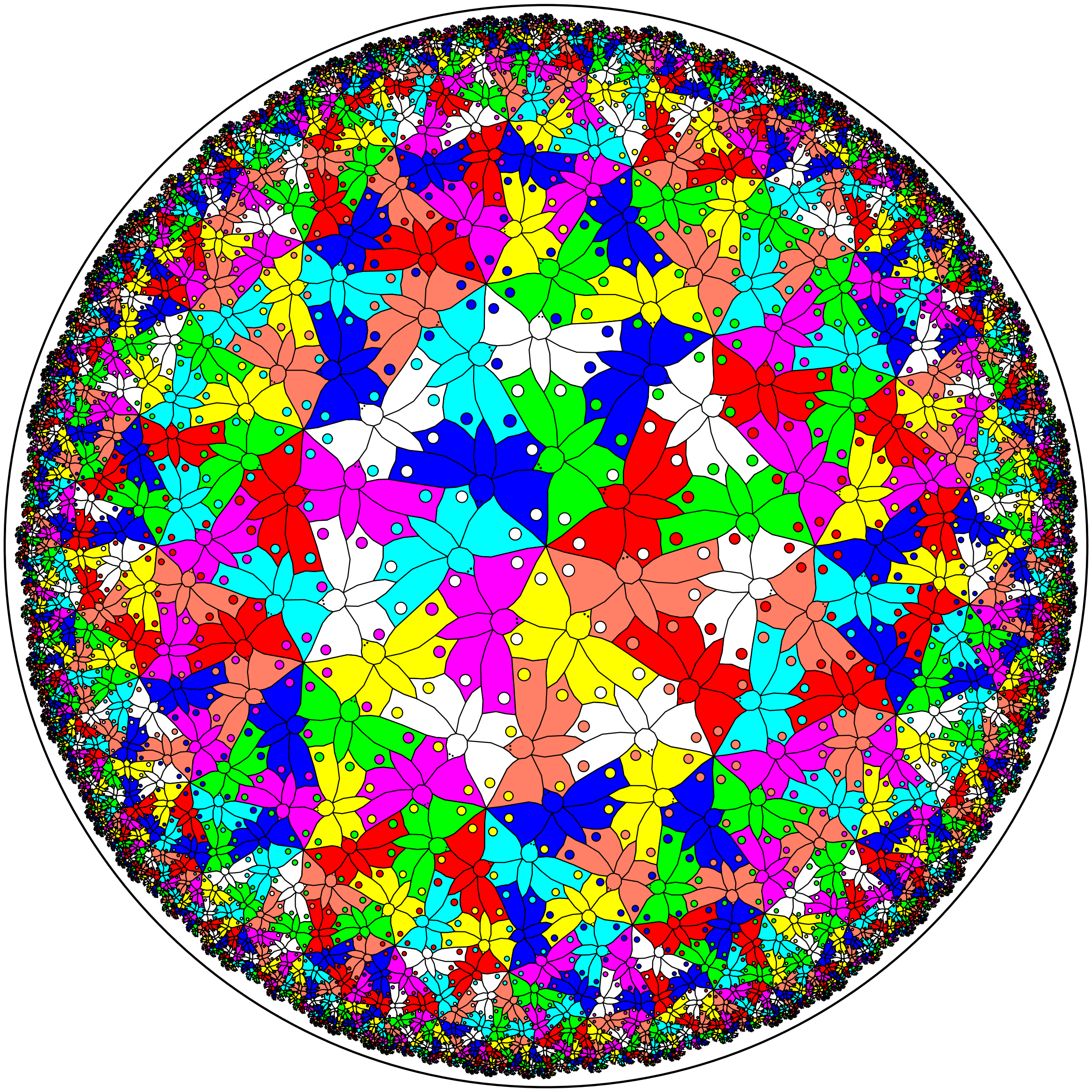
- **In the early 1960's the theory of patterns with n-color symmetry was being developed for n larger than two (the theory of 2-color symmetry had been developed in the 1930's).**
- **Previously, Escher had created many patterns with regular use of color. These patterns could now be reinterpreted in light of the new theory of color symmetry.**
- **Escher's patterns *Circle Limit II* and *Circle Limit III* exhibit 3- and 4-color symmetry respectively.**

Escher's *Circle Limit II* Pattern (3-color symmetry)

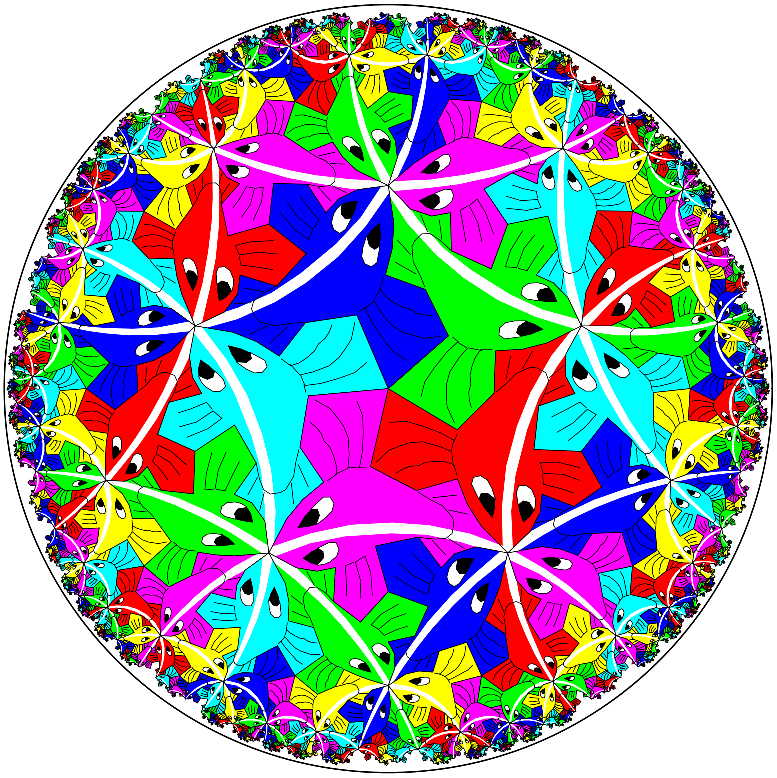
Escher's least known Circle Limit pattern.

Other Hyperbolic Patterns

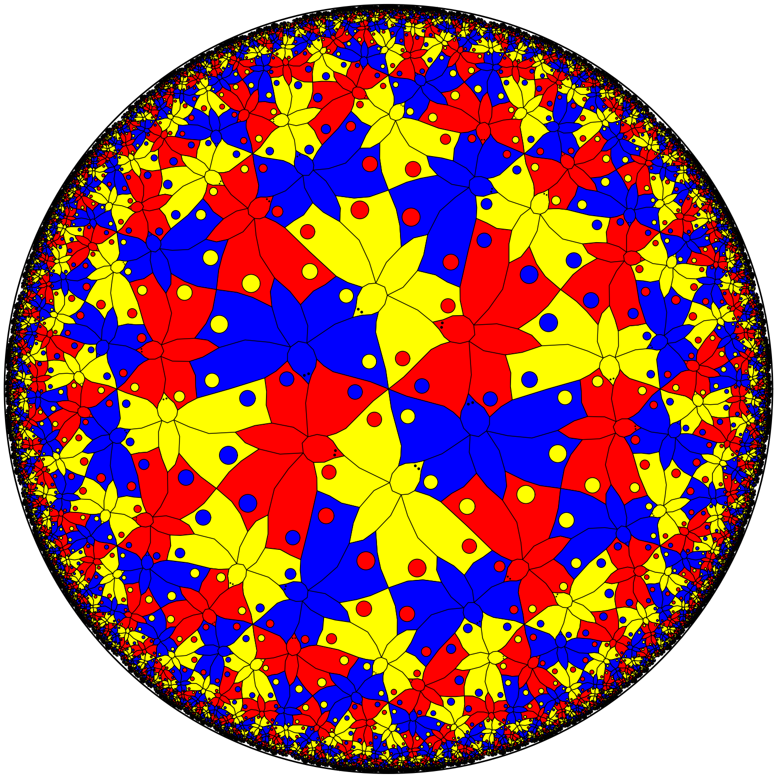
A Butterfly Pattern with Seven Butterflies at the Center (in the Art Exhibit)



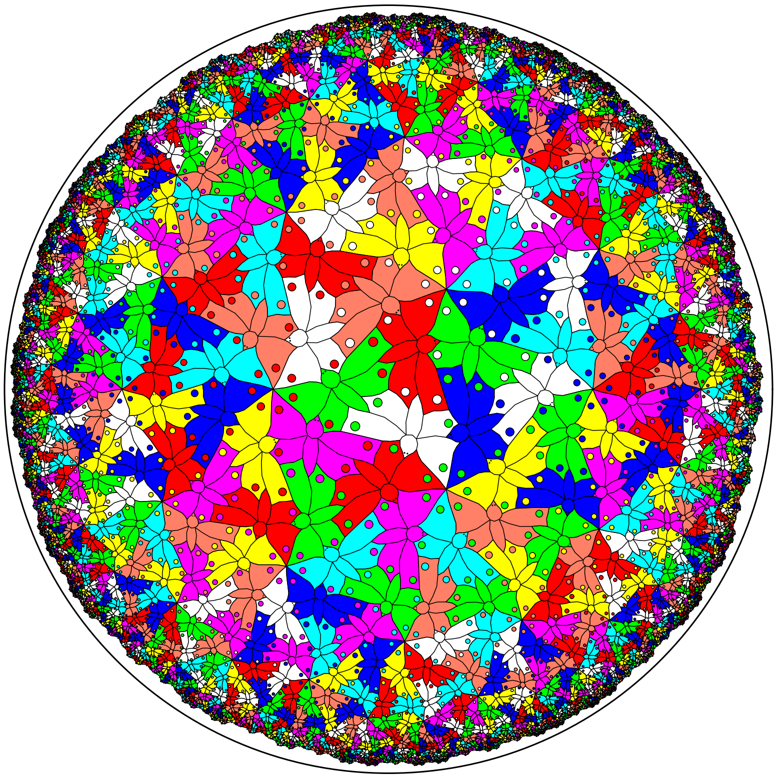
A *Circle Limit III*-like Pattern with Five Fish at the Center



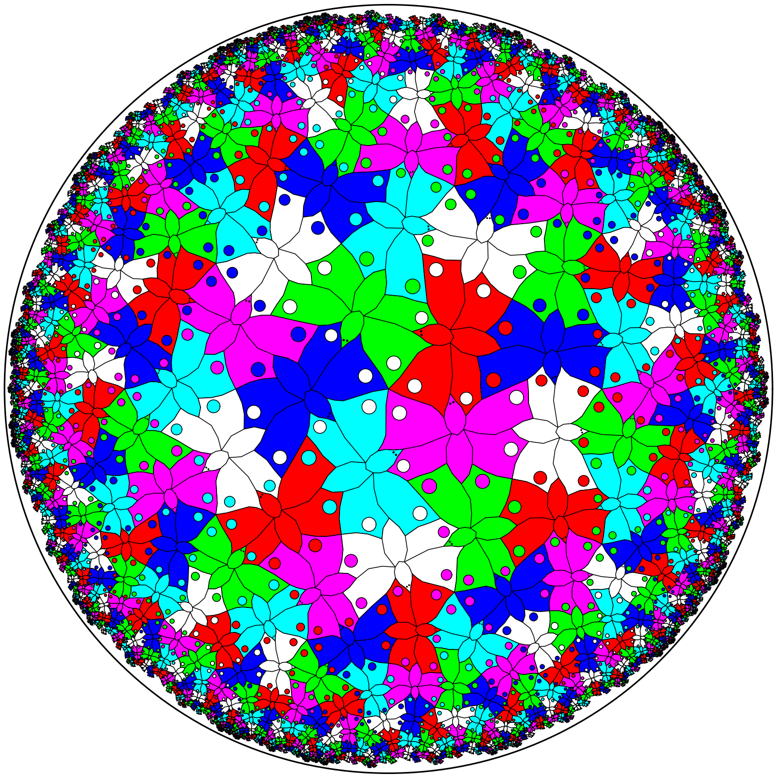
A Butterfly Pattern with Six Butterflies at the Center Center



A Butterfly Pattern with Three Butterflies at the Center Center



A Butterfly Pattern with Five Butterflies at the Center Center



Conclusions and Future Work

- **We can use repeating hyperbolic patterns to gain insight into the properties of hyperbolic geometry.**
- **We are working on a portable program that others can use to create repeating hyperbolic patterns.**