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SECTION TITLE

SYMMETRIC FISH PATTERNS ON REGULAR PERIODIC POLYHEDRA

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Abstract: *We describe each of the three infinite skew polyhedra and show how they can be decorated with angular fish patterns. The infinite skew polyhedra are regular polyhedra that repeat in three different directions in Euclidean 3-space, and are natural extensions of the Platonic solids. The fish patterns we place on them are inspired by M.C. Escher. Ordinary finite polyhedra have been decorated with patterns for centuries, but to our knowledge, this author is the first one to apply patterns to infinite polyhedra.*

Keywords: periodic polyhedra, M.C. Escher, hyperbolic geometry
MSC: 51M10, 51M20

1. INTRODUCTION

The Dutch artist M.C. Escher drew patterns on a few Platonic solids (shown in [Schattschneider 2004]). Later Doris Schattschneider and Wallace Walker decorated other Platonic solids with Escher patterns [Schattschneider 2005].

We show patterns on infinite skew polyhedra, which are natural extensions of the Platonic solids to triply periodic polyhedra. Triply periodic polyhedra are connected polyhedra with translation symmetries in three independent directions in Euclidean 3-space. Figure 1 below shows (a piece of) such a polyhedron decorated with angular fish and colored backbone lines. The infinite skew polyhedra are each composed of copies of a regular polygon, either a square or a regular hexagon. They are called “skew” since their vertex figures are skew (non-planar) polygons, in fact these polyhedra are 3-dimensional generalizations of skew polygons [Wikipedia, 2012]. We decorate each of the polyhedra with patterns of angular fish with bilateral symmetry, such as those exhibited by the Dutch artist M.C. Escher in his hyperbolic print *Circle Limit I* (a computer rendition of which can be seen in Figure 2 below). The backbone lines of the fish lie on Euclidean lines that are embedded in the polyhedra.

These polyhedra have the same topology as corresponding triply periodic minimal surfaces (TPMS); for information on TPMS's see [Schoen, 2013]. In fact each corresponding TPMS has embedded Euclidean lines that are the same as the backbone lines of the fish. Since each TPMS is a minimal surface, it has negative curvature, and thus its universal covering surface has the same large scale geometry as the hyperbolic plane. So we can theoretically transfer a pattern on an infinite skew polyhedron first to the corresponding TPMS and then from the TPMS to the hyperbolic plane. Consequently we can think of the transferred pattern in the hyperbolic plane as the *universal covering pattern* for the pattern on the polyhedron. In this two-step transformation, the backbone lines embedded in the polyhedra become hyperbolic lines in the hyperbolic plane.
In the next section, we begin with a review of regular tessellations and triply periodic polyhedra, and explain how they are related via minimal surfaces. Then we discuss fish patterns on each of the three the infinite skew polyhedra in Sections 3, 4, and 5. Finally, we indicate possibilities for other patterns on infinite polyhedra.

2. REGULAR TESSELLATIONS AND PERIODIC POLYHEDRA

We use the Schläfli symbol $\{p, q\}$ to denote the regular tessellation formed by copies of a regular p-sided polygon, or *p-gon*, with q of them meeting at each vertex. If $(p-2)(q-2) > 4$, $\{p, q\}$ is a tessellation of the hyperbolic plane, otherwise it is Euclidean or spherical. In particular $\{4, 4\}$, $\{3, 6\}$, and $\{6, 3\}$ are the familiar Euclidean tessellations, and $\{3, 3\}$, $\{3, 4\}$, $\{3, 5\}$, $\{4, 3\}$, and $\{5, 3\}$ are spherical tessellations corresponding to the Platonic solids. Figure 2 shows how Escher's *Circle Limit I* pattern is based on the $\{6, 4\}$ tessellation.

An *infinite skew polyhedron* has p-gons for faces, translation symmetries in three independent directions, and symmetry group that is transitive on flags, where

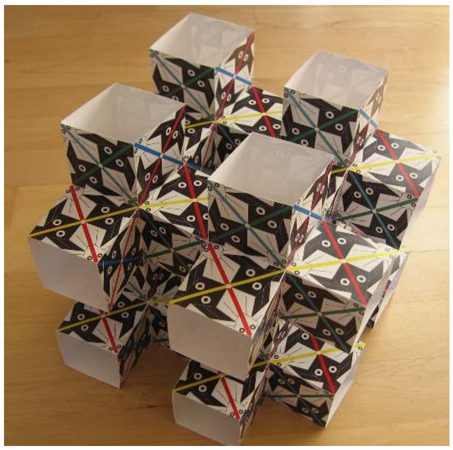

Figure 1: A piece of the $\{4,6|4\}$ polyhedron decorated with angular fish.

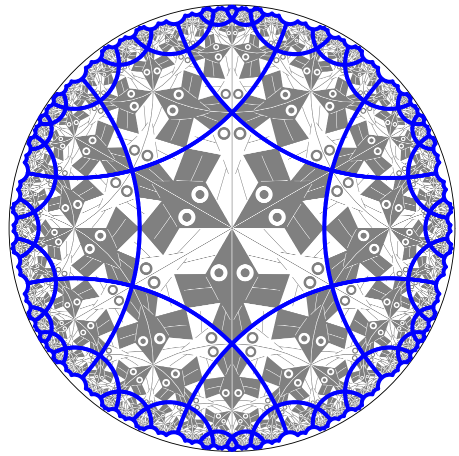

Figure 2: A computer rendition of Escher's *Circle Limit I* pattern with the {6,4} tessellation superimposed.

a flag is any triple consisting of a vertex, an edge containing that vertex, and a face containing that edge. They were discovered in 1926 by John Petrie and H.S.M. Coxeter who designated them by the extended Schläfli symbol $\{p,q|n\}$, indicating that there are q p-gons around each vertex and n-gonal holes [Wikipedia, 2012]. There are exactly three of them, $\{4,6|4\}$ (shown in Figure 1), $\{6,4|4\}$, and $\{6,6|3\}$. We discuss fish patterns on them in the next three sections.

3. A FISH PATTERN ON THE $\{4,6|4\}$ POLYHEDRON

One can see by examining Figure 1 that the $\{4,6|4\}$ polyhedron is based on the cubic lattice in 3-space. In fact it divides 3-space into two complementary congruent solids. Each of the solids is composed of "hub" cubes with "strut" cubes on each of its faces, with each strut cube connecting two hub cubes.

In the pattern of Figure 1, the backbones of the fish in a horizontal plane lie along parallel red lines or parallel yellow lines that are perpendicular to the red lines. Similarly for planes facing the lower left, fish backbones lie along green and cyan lines; and for planes facing the lower right, the backbones lie along blue and magenta lines. Also there are two kinds of vertices: those where red, blue, and green backbone lines intersect (at 60 degree angles), and those where cyan, magenta, and yellow lines intersect. Figure 3 shows a close-up view of the latter

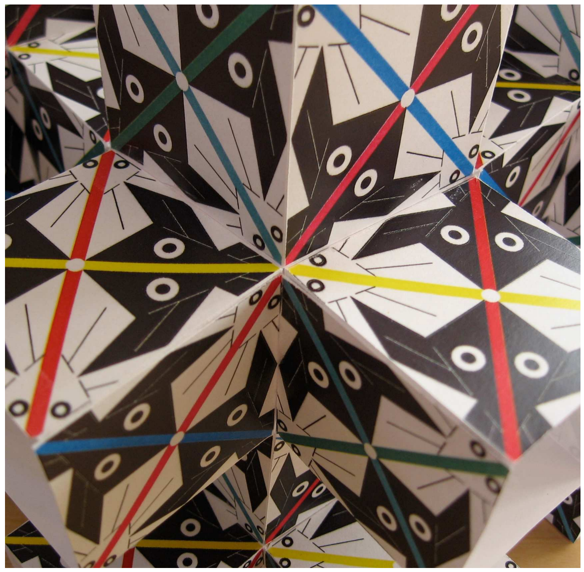

Figure 3: A vertex view of the fish pattern on the $\{4, 6|4\}$ polyhedron.

kind of vertex. Figure 4 shows the universal covering pattern of Figure 1, including the colored backbone lines. The backbone lines of the fish in Figures 1 and 3 are closely related to the Schwarz P-surface (the P stands for Primative), since they all lie on that surface (for information on the P-surface, a TPMS, see [Schoen, 2013]). Figure 5 shows the P-surface with the embedded lines. One can see that they are the same as the backbone lines of Figure 1. In fact the backbone lines form skew rhombi, which can be seen in Figures 1 and 3. If these rhombi are spanned by soap films, i.e. minimal surfaces, one obtains the Schwarz P-surface.

4. A FISH PATTERN ON THE $\{6, 4|4\}$ POLYHEDRON

The $\{6, 4|4\}$ polyhedron is the dual of the $\{4, 6|4\}$ polyhedron in which each vertex is replaced by a hexagon. The $\{4, 6|4\}$ polyhedron is based on the bi-truncated, cubic, space-filling tessellation by truncated octahedra. The $\{6, 4|4\}$ polyhedron divides space into two sets of truncated octahedra, the truncated octahedra of each set being connected by their square faces. Figure 6 shows the backbone lines of the fish pattern on the $\{6, 4|4\}$ polyhedron, which are the same lines as the backbone lines of the fish in Figure 1, which is not surprising since the polyhedra are duals to each other. Consequently the TPMS corresponding to the $\{6, 4|4\}$ polyhedron

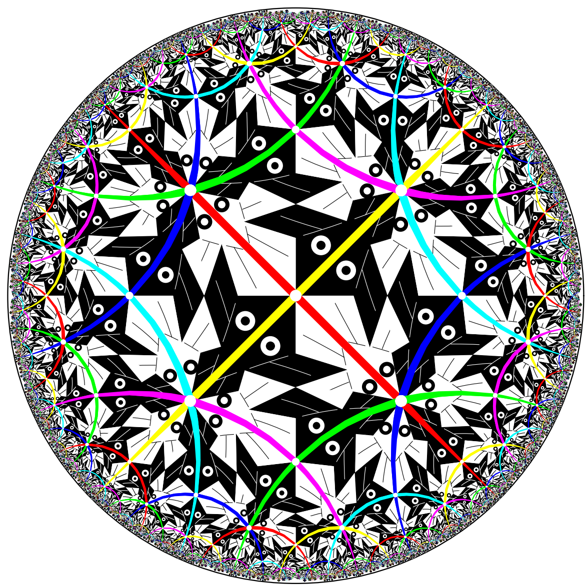

Figure 4: A hyperbolic plane pattern corresponding to Figure 1.

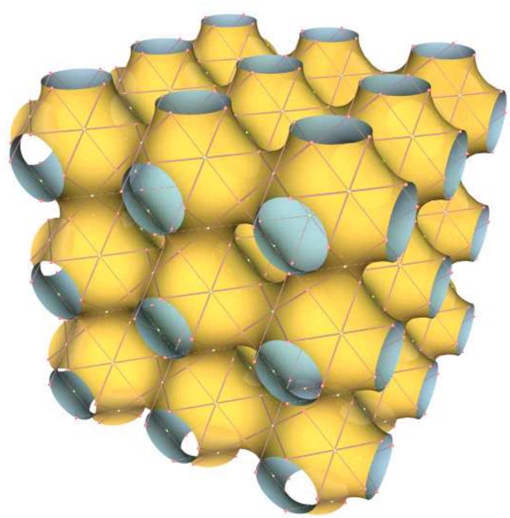

Figure 5: Schwarz's P-surface showing embedded lines.

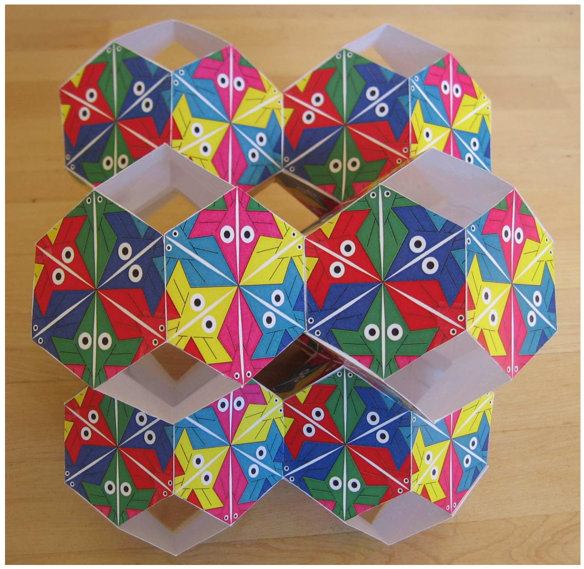

Figure 6: A piece of the $\{6,4|4\}$ polyhedron decorated with angular fish.

is the same as that of the $\{4,6|4\}$ polyhedron: the Schwarz P-surface. Figure 7 shows a top view of the polyhedron of Figure 6, in which some of the backbone lines become more apparent. Figure 8 shows the "hyperbolic covering pattern" of Figures 6 and 7.

4. A FISH PATTERN ON THE $\{6,6|3\}$ POLYHEDRON

The self-dual $\{6,6|3\}$ polyhedron may be the most difficult to understand. It is formed from truncated tetrahedra with their triangular faces removed. Such "missing" triangular faces from four truncated tetrahedra are then placed in a tetrahedral arrangement (around a small invisible tetrahedron). Figure 9 shows a side view of a $\{6,6|3\}$ polyhedron decorated with angular fish. Figure 10 shows a top view looking down at one of the vertices (where six hexagons meet). Figure 11 shows the corresponding universal covering pattern based on the $\{6,6\}$ tessellation.

The corresponding TPMS to the $\{6,6|3\}$ polyhedron is the Schwarz D-surface (the D stands for Diamond; see [Schoen, 2013]). The Schwarz D-surface divides space into two congruent parts, each with the shape of a thickened diamond lattice.

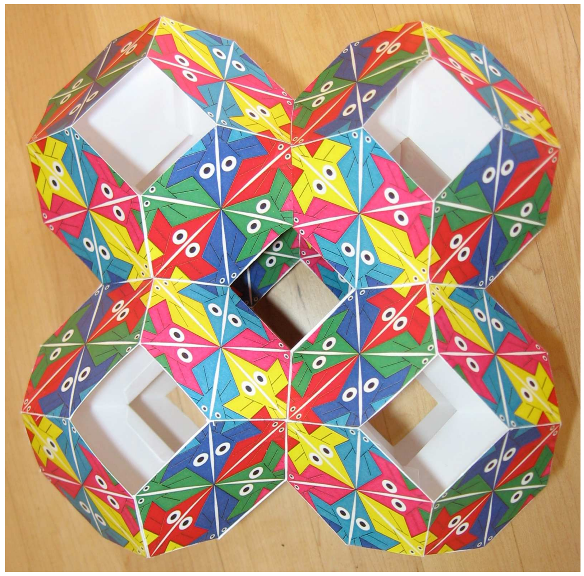

Figure 7: A top view of the $\{6, 4|4\}$ polyhedron decorated with angular fish.

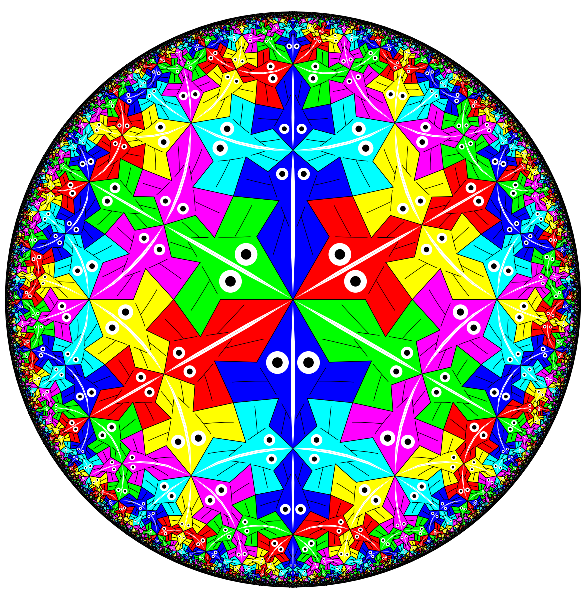

Figure 8: The pattern in the hyperbolic plane corresponding to Figure 6.

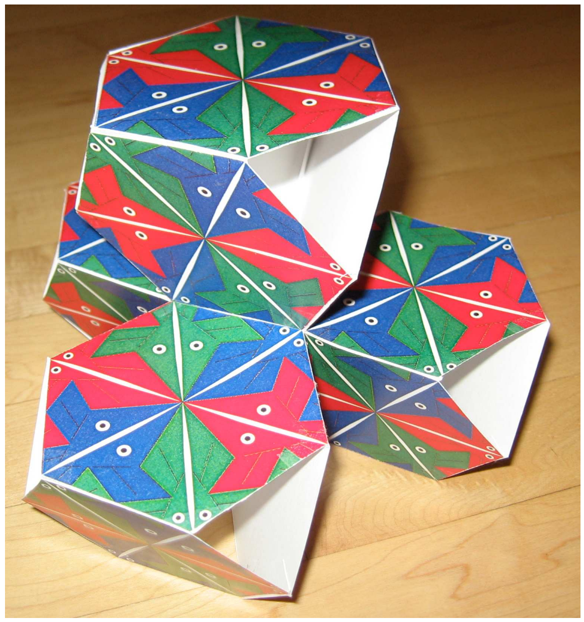

Figure 9: A piece of the $\{6,6|3\}$ polyhedron decorated with angular fish.

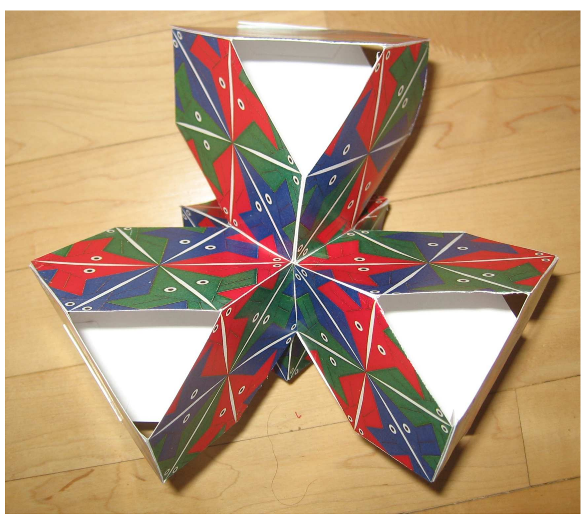

Figure 10: A top view of a pattern of fish on the $\{6,6|3\}$ polyhedron shown in Figure 9.

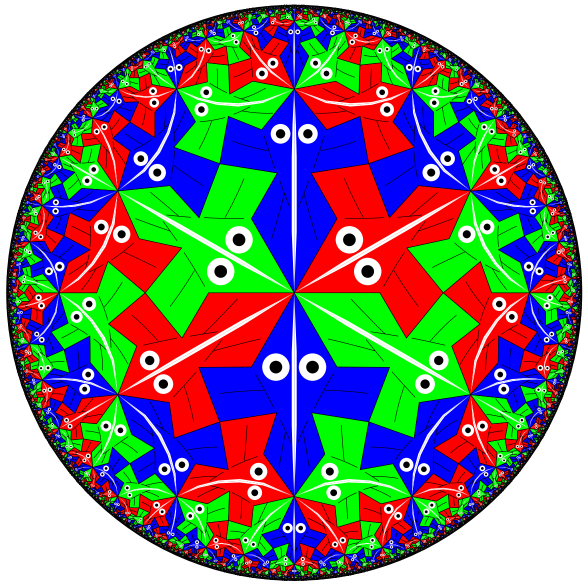

Figure 11: The pattern in the hyperbolic plane corresponding to Figures 9 and 10.

The backbone lines of Figure 9 are embedded lines in the Schwarz D-surface. As with the $\{4, 6|4\}$ and $\{6, 4|4\}$ polyhedra, the backbone lines form skew rhombi. And as before, if the rhombi of the $\{6, 6|3\}$ pattern are spanned by soap films, i.e. minimal surfaces, one obtains the Schwarz D-surface in this case. Figure 12 shows a piece of the Schwarz D-surface within a rhombic dodecahedron, and since rhombic dodecahedra fill space, one can obtain the entire Schwarz D-surface.

3. CONCLUSION AND FUTURE WORK

We have shown fish patterns on each of the infinite skew polyhedra. It would be interesting to place other patterns on these polyhedra too. Although it has been known for 85 years that there are only three infinite skew polyhedron, the more general uniform triply periodic polyhedra have not been classified, but a number of examples are known. It would be challenging not only to place mathematically and artistically interesting patterns on the known polyhedra, but even more challenging to discover new polyhedra.

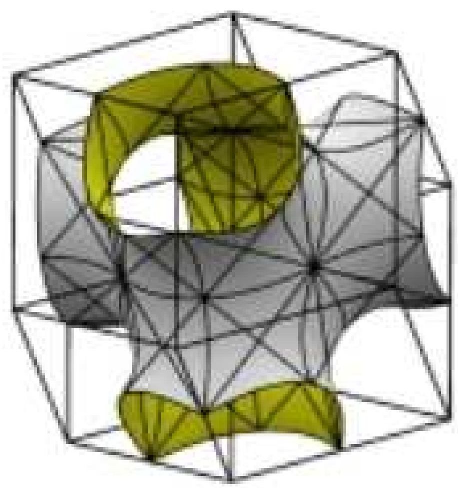

Figure 12: A piece of the Schwarz D-surface within a rhombic dodecahedron.

REFERENCES

Schoen, A.H., (2013) *Web Site for Triply Periodic Minimal Surfaces*, `http://schoengeometry.com/e_tpms.html`

Wikipedia (2012) *Wikipedia entry for Infinite skew polyhedron* `http://en.wikipedia.org/wiki/Infinite_skew_polyhedron`

Schattschneider, D., (2004) *M.C. Escher: Visions of Symmetry*, 2nd Ed., New York, Harry N. Abrams, Inc., 2004, 384 pp. ISBN 0-8109-4308-5

Schattschneider, D., (2005) and W. Walker, *M.C. Escher Kaleidocycles,* California, Pomegranate Press, 2005, 48 pp. ISBN 0764931105