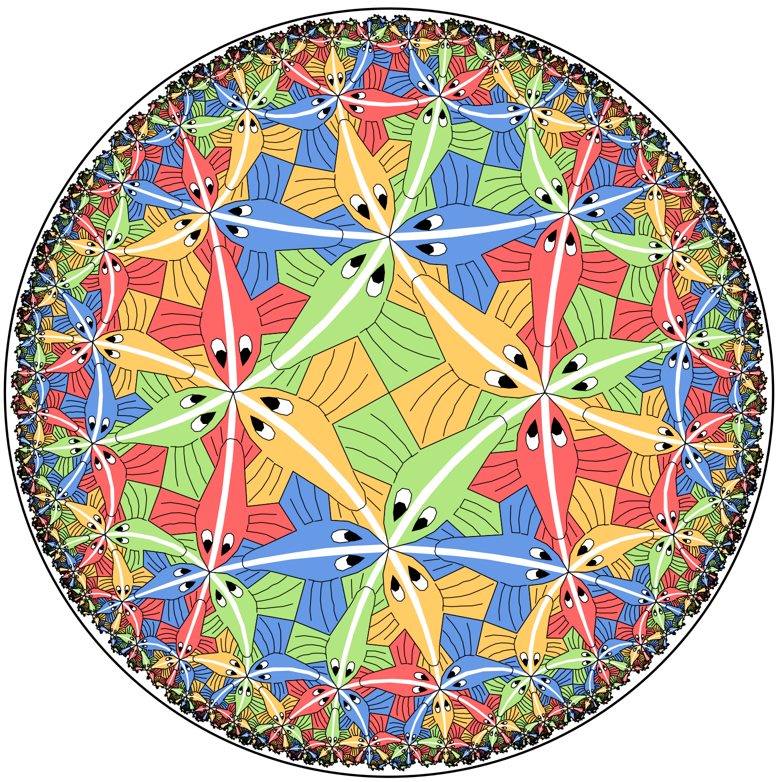
The Interplay Between Hyperbolic Symmetry and History

Douglas Dunham
Department of Computer Science
University of Minnesota, Duluth
Duluth, MN 55812-3036, USA
E-mail: `ddunham@d.umn.edu`
Web Site: `http://www.d.umn.edu/~ddunham/`



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Introduction

There are 3 “Classical Geometries”:

- **the Euclidean plane**
- **the sphere**
- **the hyperbolic plane**

The first two have been known since antiquity

Hyperbolic geometry has been known for slightly less than 200 years.

Hyperbolic Geometry Discovered Independently in Early 1800's by:

- **Bolyai János**
- **Carl Friedrich Gauss (did not publish results)**
- **Nikolas Ivanovich Lobachevsky**

Why the Late Discovery of Hyperbolic Geometry?

- **Euclidean geometry seemed to represent the physical world.**
- **Unlike the sphere was no smooth embedding of the hyperbolic plane in familiar Euclidean 3-space.**

Euclid and the Parallel Axiom

- **Euclid realized that the parallel axiom, the Fifth Postulate, was special.**
- **Euclid proved the first 28 Propositions in his Elements without using the parallel axiom.**

Attempts to Prove the Fifth Postulate

- **Adrien-Marie Legendre (1752–1833) made many attempts to prove the Fifth Postulate from the first four.**
- **Girolamo Saccheri (1667–1733) assumed the negation of the Fifth Postulate and deduced many results (valid in hyperbolic geometry) that he thought were absurd, and thus "proved" the Fifth Postulate was correct.**

New Geometries

- **A negation of the Fifth Postulate, allowing more than one parallel to a given line through a point, leads to another geometry, hyperbolic, as discovered by Bolyai, Gauss, and Lobachevsky.**
- **Another negation of the Fifth Postulate, allowing no parallels, and removing the Second Postulate, allowing lines to have finite length, leads to elliptic (or spherical) geometry, which was also investigated by Bolyai.**

A Second Reason for Late Discovery

- **As mentioned before, there is no smooth embedding of the hyperbolic plane in familiar Euclidean 3-space (as there is for the sphere). This was proved by David Hilbert in 1901.**
- **Thus we must rely on "models" of hyperbolic geometry - Euclidean constructs that have hyperbolic interpretations.**

The Beltrami-Klein Model of Hyperbolic Geometry

- **Hyperbolic Points: (Euclidean) interior points of a bounding circle.**
- **Hyperbolic Lines: chords of the bounding circle (including diameters as special cases).**
- **Described by Eugenio Beltrami in 1868 and Felix Klein in 1871.**

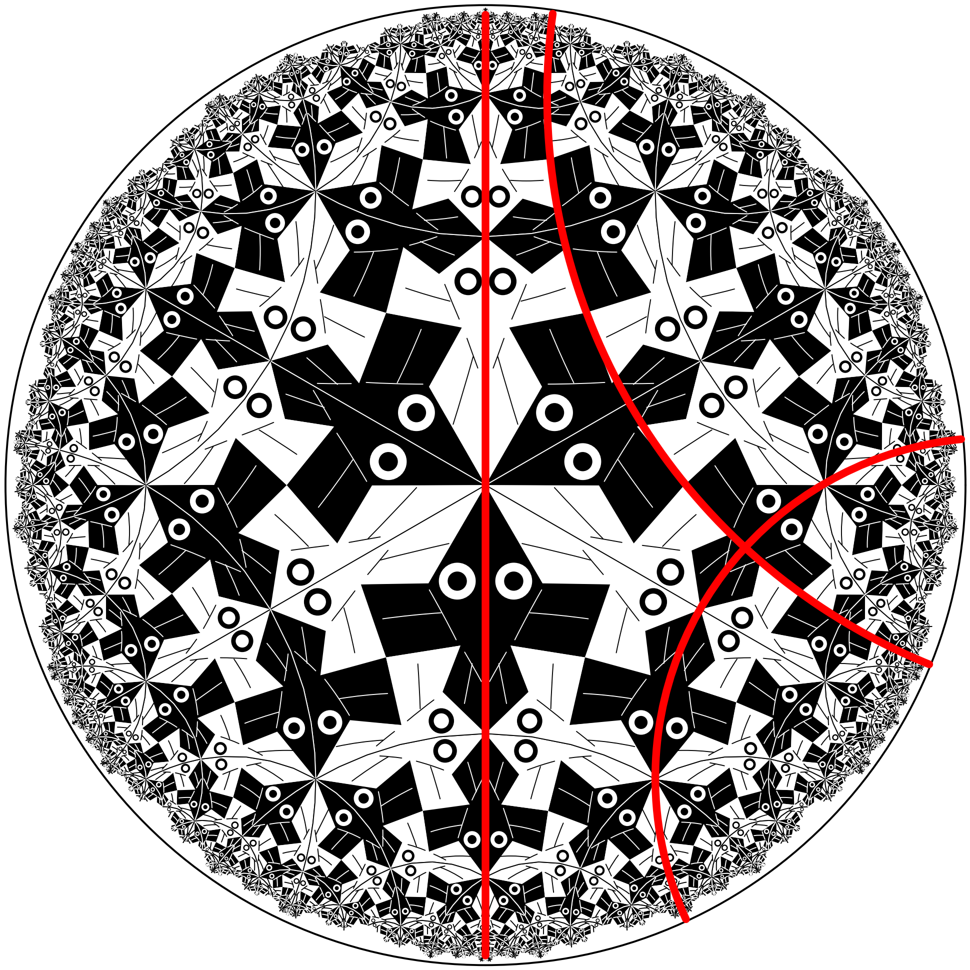
Consistency of Hyperbolic Geometry

- **Hyperbolic Geometry is at least as consistent as Euclidean geometry, for if there were an error in hyperbolic geometry, it would show up as a Euclidean error in the Beltrami-Klein model.**
- **So then hyperbolic geometry could be the “right” geometry and Euclidean geometry could have errors!?!?**
- **No, there is a model of Euclidean geometry within 3-dimensional hyperbolic geometry — they are equally consistent.**

Henri Poincaré's Models of Hyperbolic Geometry

- **The circle model (described next).**
- **The upper half-plane model:**
  - **Hyperbolic points: points in the xy-plane with positive y-coordinate.**
  - **Hyperbolic lines: upper semi-circles whose center is on the x-axis (including vertical half-lines as special cases).**
  - **Used by M.C. Escher in his "line-limit" patterns.**
- **Both models are *conformal*: the hyperbolic measure of angles is the same as their Euclidean measure.**

Poincaré Circle Model of Hyperbolic Geometry



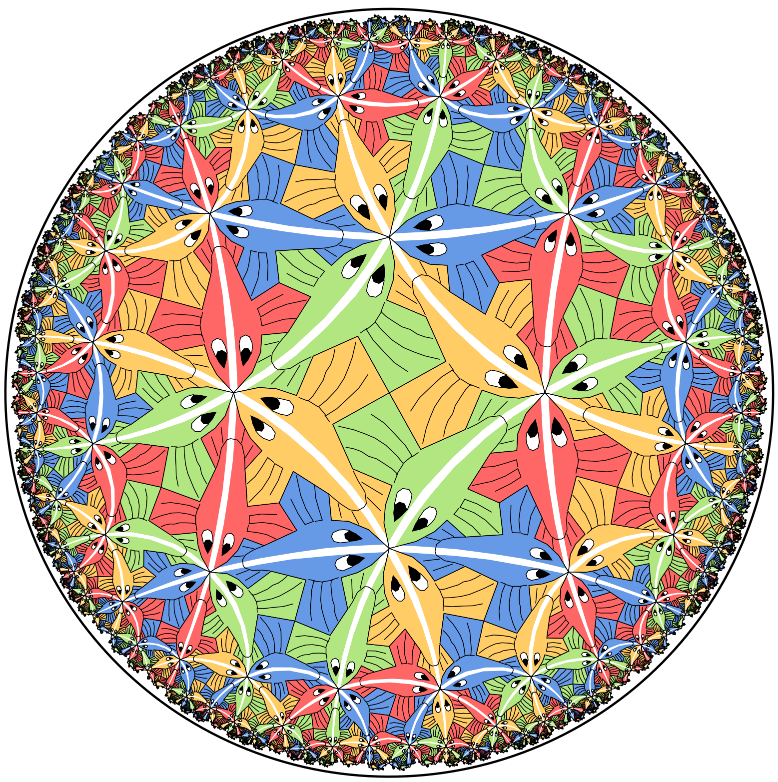
- **Points: points within the (unit) bounding circle**
- **Lines: circular arcs perpendicular to the bounding circle (including diameters as a special case)**
- **Attractive to Escher and other artists because it is represented in a finite region of the Euclidean plane, so viewers could see the entire pattern, and is conformal, so that copies of a motif in a repeating pattern retained their same approximate shape.**

Relation Between Poincaré and Beltrami-Klein Models

- **A chord in the Beltrami-Klein model represents the same hyperbolic line as the orthogonal circular arc with the same endpoints in the Poincaré circle model.**
- **The models measure distance differently, but equal hyperbolic distances are represented by ever smaller Euclidean distances toward the bounding circle in both models.**

Repeating Patterns

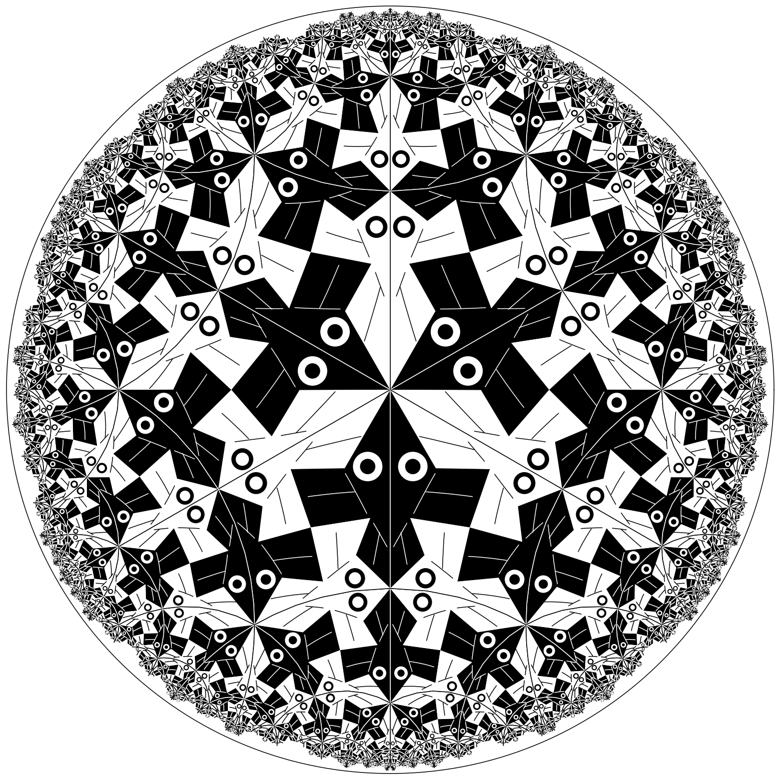
- **A *repeating pattern* in any of the three classical geometries is a pattern made up of congruent copies of a basic subpattern or *motif*.**
- **The copies of the motif are related by *symmetries* — isometries (congruences) of the pattern which map one motif copy onto another.**
- **A fish is a motif for Escher's *Circle Limit III* pattern below if color is disregarded.**



Symmetries in Hyperbolic Geometry

- **A reflection is one possible symmetry of a pattern.**
- **In the Poincaré disk model, reflection across a hyperbolic line is represented by inversion in the orthogonal circular arc representing that hyperbolic line.**
- **As in Euclidean geometry, any hyperbolic isometry, and thus any hyperbolic symmetry, can be built from one, two, or three reflections.**
- **For example in either Euclidean or hyperbolic geometry, one can obtain a rotation by applying two successive reflections across intersecting lines, the angle of rotation being twice the angle of intersection.**

Examples of Symmetries — Escher's *Circle Limit I*



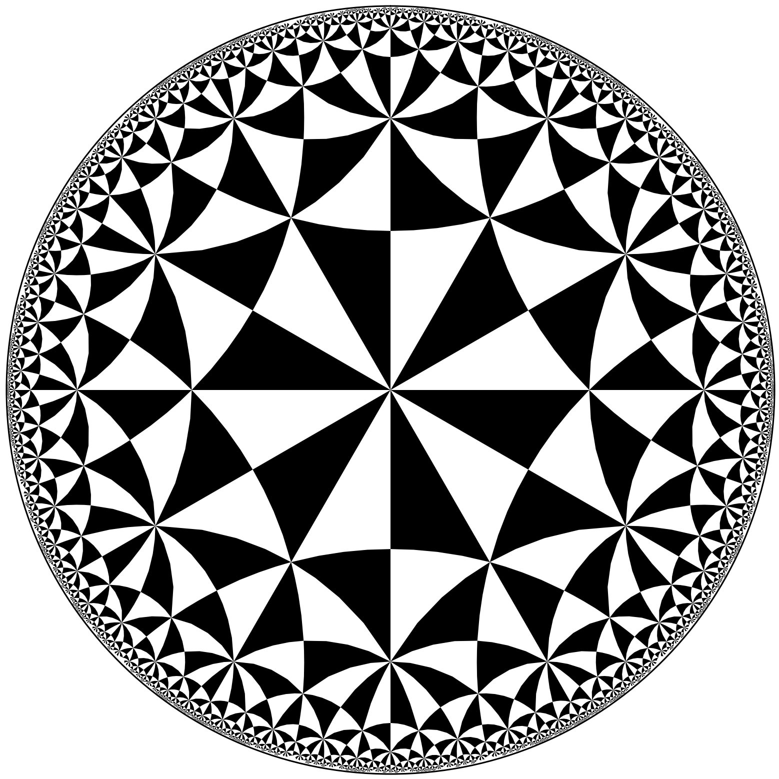
Repeating Patterns and Hyperbolic Geometry

- **Repeating patterns and their symmetry are necessary to understand the hyperbolic nature of the different models of hyperbolic geometry.**
- **For example, a circle containing a few chords may just be a Euclidean construction and not represent anything hyperbolic in the Beltrami-Klein model.**
- **On the other hand, if there is a repeating pattern within the circle whose motifs get smaller as one approaches the circle, then we may be able to interpret it as a hyperbolic pattern.**

Symmetry, Discovery, Aesthetics, and New Knowledge

- **Probably the most important example of hyperbolic symmetry leading to something new was Escher's discovery of the Poincar disk model as a solution to his long-sought "circle limit" problem: how to construct a repeating pattern whose motifs get infinitely small toward the edge of a bounding circle.**
- **In a "circle limit" pattern, the complete infinite pattern could be captured within a finite area, unlike the "point limit" and "line limit" patterns Escher had previously designed.**
- **The Canadian mathematician H.S.M. Coxeter sent Escher a reprint of a paper containing such a pattern of curvilinear triangles.**
- **Escher recounted that "gave me quite a shock" since the apparent visual symmetries of the pattern instantly showed him the solution to his problem.**

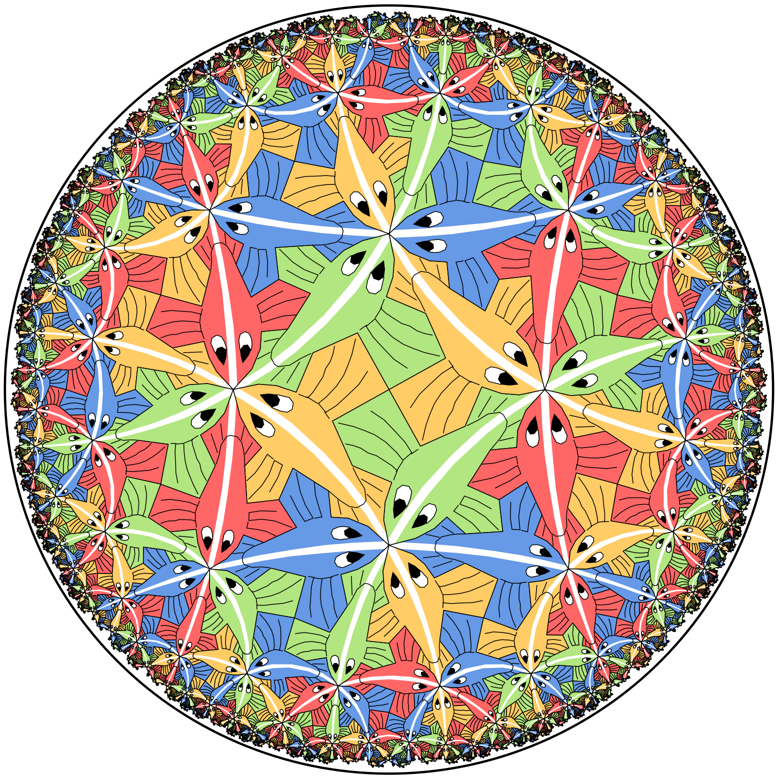
Coxeter's Pattern that Inspired Escher



Escher Inspiring Coxeter and Dunham

- **In turn, Escher's "Circle Limit" patterns inspired Coxeter to analyze them.**
- **Coxeter determined the symmetry groups of Escher's *Circle Limit II*, *Circle Limit III*, and *Circle Limit IV* as [$3^+, 8$], ($4, 3, 3$), and [$4^+, 6$] respectively.**
- **Later, Dunham, inspired by both Escher and Coxeter determined the symmetry group of *Circle Limit I* to be $cmm_{3,2}$, a special case of $cmm_{\frac{p}{2},\frac{q}{2}}$ (which is generated by reflections across the sides of a rhombus with angles $2\pi/p$ and $2\pi/q$, and a 180 degree rotation about its center). When $p = q = 4$, we get the Euclidean "wallpaper" group $cmm = cmm_{2,2}$.**
- **Coxeter also determined that backbone arcs of *Circle Limit III* made an angle of ω with the bounding circle, where $\cos(\omega) = \sqrt{\frac{3\sqrt{2}-4}{8}}$**

Escher's *Circle Limit III* — Note Backbone Lines

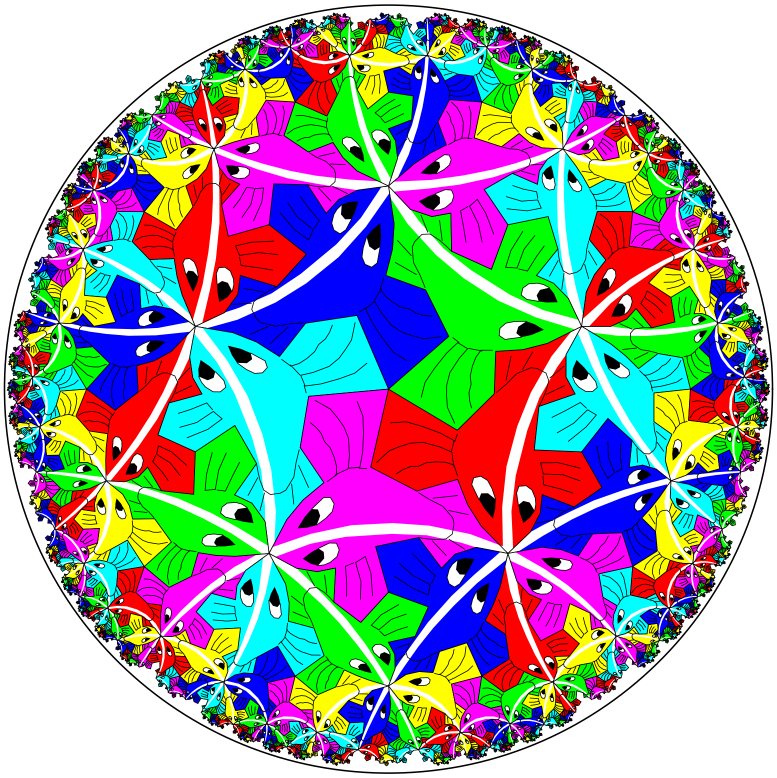


Escher and Coxeter Inspiring Dunham

- **Several years ago I was inspired by Escher and an editor of the *Mathematical Intelligencer* to conceive of a generalization of Escher's *Circle Limit III* pattern.**
- **Allowing p fish to meet at right fin tips, q fish to meet at left fin tips, and r fish to meet nose-to-nose, I denote the resulting pattern (p, q, r), so *Circle Limit III* would be called $(4, 3, 3)$ in this notation.**
- **Luns Tee and Dunham also determined that backbone arcs of of a (p, q, r) pattern made an angle of ω with the bounding circle, where**

$$\cos(\omega) = \frac{\sin(\frac{\pi}{2r})\ (\cos(\frac{\pi}{p}) - \cos(\frac{\pi}{q}))}{\sqrt{\cos(\frac{\pi}{p})^2 + \cos(\frac{\pi}{q})^2 + \cos(\frac{\pi}{r})^2 + 2\cos(\frac{\pi}{p})\cos(\frac{\pi}{q})\cos(\frac{\pi}{r}) - 1)}}$$

A (5,3,3) Pattern



Reinterpretation of Geometric Knowledge

- **Lobachevsky and probably Gauss viewed hyperbolic geometry in isolation.**
- **Bolyai also considered elliptic geometry as another possible non-Euclidean geometry, thus becoming the first to consider the three classical geometries, Euclidean, hyperbolic, and elliptic, as a related set distinguished by their parallel properties.**
- **The classical geometries also have constant curvature: zero curvature for the Euclidean plane, positive curvature for elliptic geometry, and negative curvature for hyperbolic geometry.**
- **Gauss considered more general 2-dimensional surfaces in which the curvature could also vary from point to point, so the classical geometries were then just special cases.**
- **In the mid-1800's, Riemann extended the concept of surfaces to n-dimensional manifolds, so surfaces in turn became just a special case of manifolds (of dimension two).**

Relative Consistency of Euclidean and Hyperbolic Geometry

- **Beltrami's interpretation of his model of hyperbolic geometry as a set of Euclidean constructs proved that hyperbolic geometry was just as consistent as Euclidean geometry.**
- **Conversely, it was shown later that horospheres in 3-dimensional hyperbolic space could be interpreted as having the same structure as the Euclidean plane, thus proving that the geometries were equally consistent.**

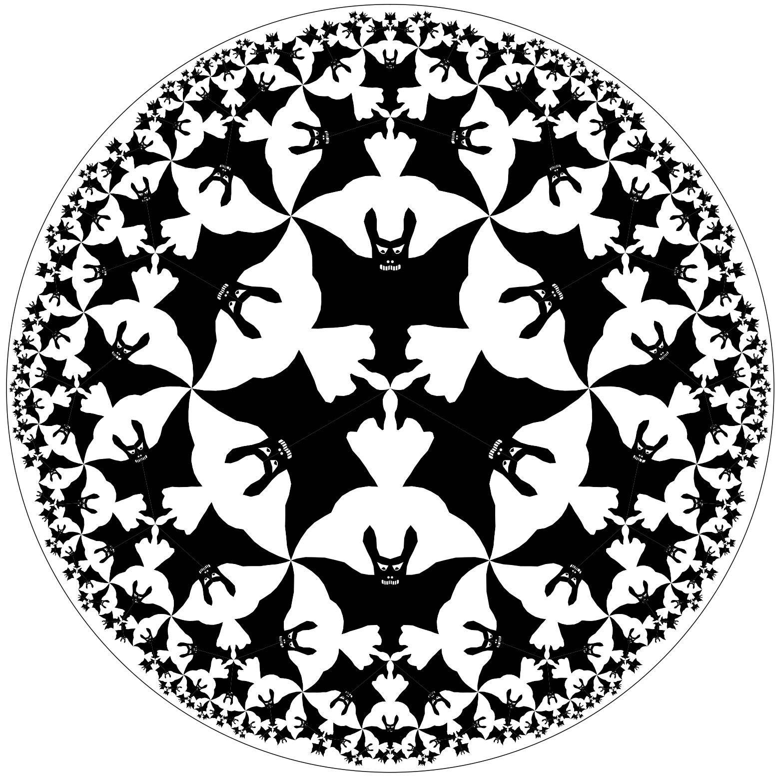
Color Symmetry

- **In the early 1960's the theory of patterns with n-color symmetry was being developed for n larger than two (the theory of 2-color symmetry had been developed in the 1930's).**
- **Previously, Escher had created many patterns with regular use of color. These patterns could now be reinterpreted in light of the new theory of color symmetry.**
- **Escher's patterns *Circle Limit II* and *Circle Limit III* exhibit 3- and 4-color symmetry respectively.**

Escher's *Circle Limit II* Pattern (3-color symmetry)

Escher's least known Circle Limit pattern.

Escher's *Circle Limit IV* Pattern



- **Only motif Escher rendered in all three classical geometries**
- **This pattern inspired Australian polymath Tony Bomford to create hyperbolic hooked rugs.**

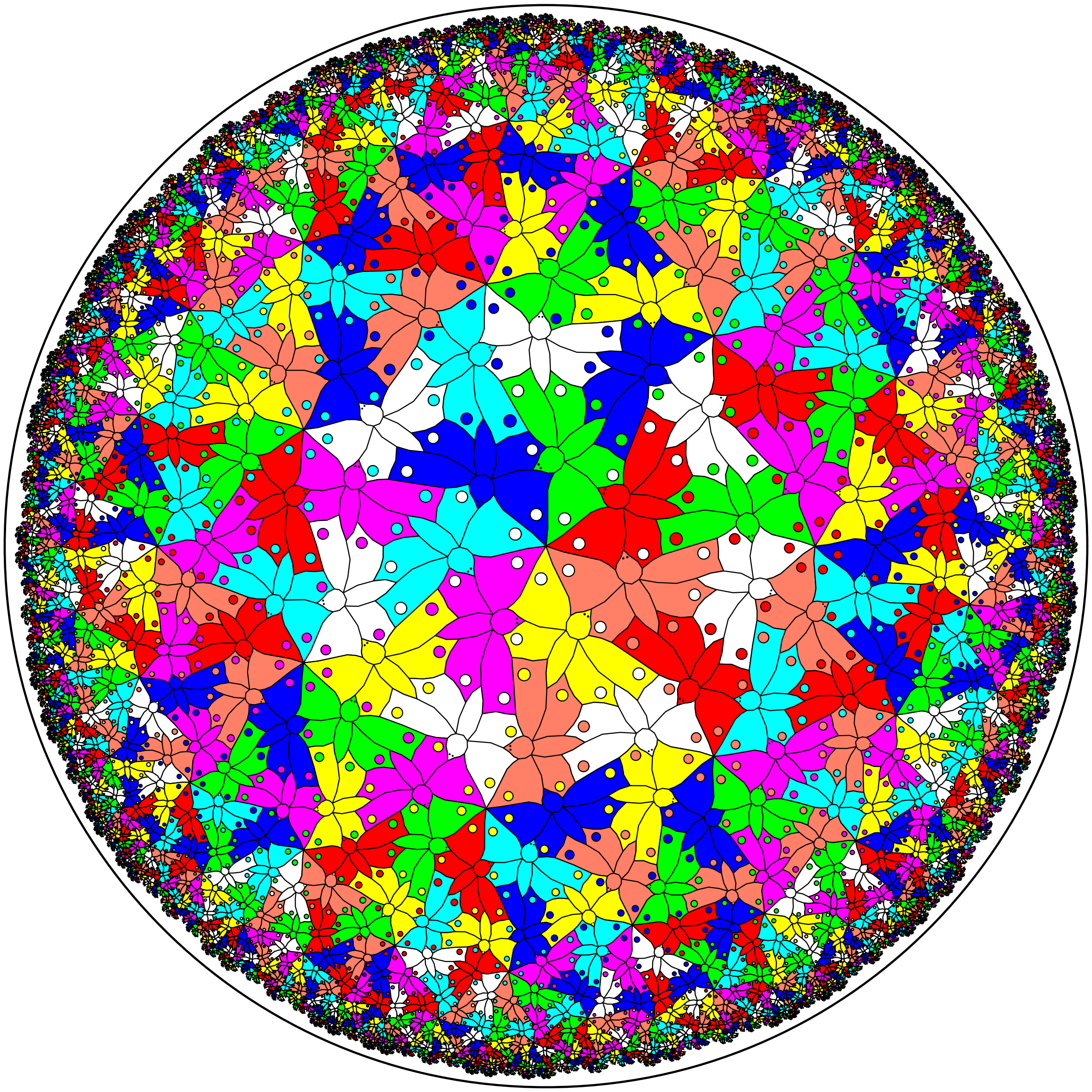
Integration of Knowledge to Create Repeating Hyperbolic Patterns

- **Repeating hyperbolic patterns were first created a little more than 100 years ago. These were abstract patterns like the pattern of curvilinear triangles used by Coxeter that inspired Escher.**
- **Creating these patterns mostly required the knowledge of Euclidean constructions.**
- **To create aesthetically pleasing hyperbolic patterns it is necessary to have good artistic knowledge also.**
- **Escher became the first person to create such patterns because he had both the mathematical and the artistic knowledge.**

Creating Hyperbolic Patterns Using a Computer

- **Creating repeating hyperbolic patterns by hand is tedious and time consuming, since unlike Euclidean repeating patterns in which the motifs are all the same size, the motifs vary in size and must be created individually.**
- **About 30 years ago it occurred to me that computers could be used to draw hyperbolic patterns but they had to be programmed to do so.**
- **Thus a third discipline, computer science, needed to be integrated into the process.**
- **My students and I were successful in this endeavor, not only re-creating Escher's circle limit patterns, but other patterns as well.**

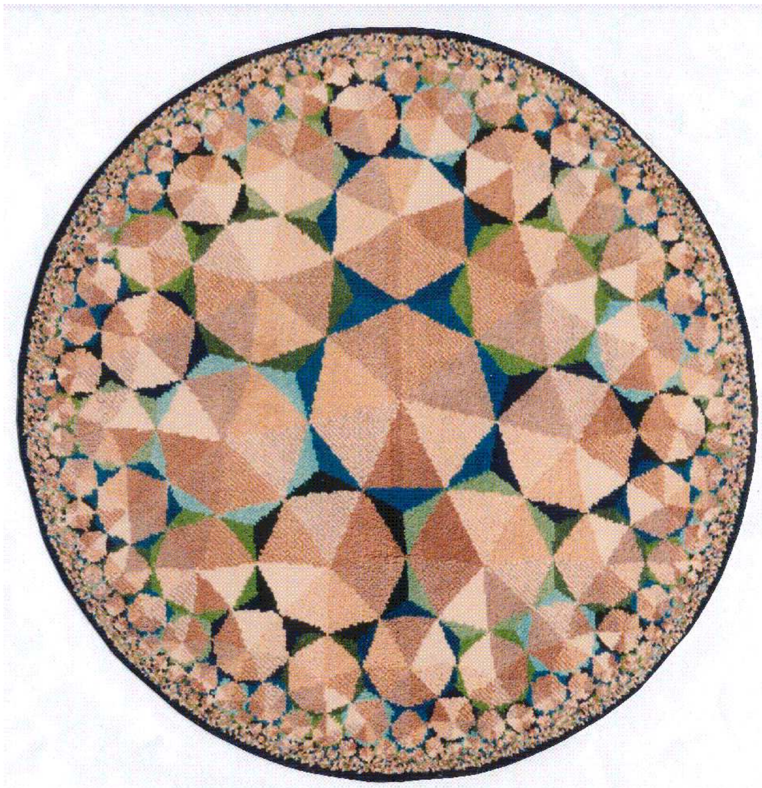
Another Hyperbolic Pattern



Creating More Hyperbolic Art Using Computers

- **The computer programs that my students and I designed have also been used to draw templates that could be used by artists using other media.**
- **Irene Rousseau was the first artist to take advantage of hyperbolic pattern templates, creating several glass mosaic patterns.**
- **Later Mary Williams made a quilt with a hyperbolic pattern, inspired by one of my computer drawings.**
- **Currently I am working with a hooked rug maker to create a new hyperbolic pattern related to those made by the Australian rug maker Tony Bomford.**

Tony Bomford's Rug Number 17



Integration of the Cerebral Hemispheres in Creating Patterns

- **The process of creating repeating hyperbolic patterns via computer requires both left-brain and right-brain thinking.**
- **The left-brain functions of logic and mathematics are necessary to create the computer programs to draw the patterns.**
- **The computer program also makes fundamental use of the 3-dimensional Weierstrass model of hyperbolic geometry.**
- **Thus the right-brain functions of geometric insight, artistic appreciation, and 3D conceptualization are also needed.**
- **Consequently, the creation of these patterns requires an integrated and balanced use of both the left-brain and right-brain hemispheres.**

Future Work

- **Extend the classification of families of Escher patterns such as his "Angels and Devils" patterns (in each of the three classical geometries), and the proposed (p, q, r) generalizations of *Circle Limit III*.**
- **Find an algorithm for computing the minimum number of colors needed for a pattern, given the definition of its symmetry group (ignoring color).**
- **Continue to inspire other artists to create hyperbolic patterns in their media.**