Math 3280 Numerical methods/IVP review solutions.
Please let me know if you find a typo.

(a) Solve the initial value problem $x'' + 2x' + 5x = t$, $x(0) = 0$, $x'(0) = 0$, and find the exact value of $x(2)$.

Solution: This could be done with the Laplace transform or undetermined coefficients.
With undetermined coefficients, we first find the homogeneous (or complementary) solution x_c. To do this, we factor the characteristic polynomial

$$r^2 + 2r + 5 = (r - (-1 + 2i))(r - (-1 - 2i)) = 0$$

using the quadratic formula. Since the real part of this pair of complex conjugate roots is -1, and the imaginary part is ± 2, we know that

$$x_c = C_1 e^{-t} \cos(2t) + C_2 e^{-t} \sin(2t)$$

Next we find the particular solution. Since the right-hand side of the ODE has the nonhomogeneous term t, we will use

$$x_p = At + B$$

unless there is some overlap with x_c. Since there is no overlap here, we do not shift x_p by any power of t.
To determine A and B, we plug x_p and its derivatives $x_p' = A$, $x_p'' = 0$ into the ODE to get

$$0 + 2(A) + 5(At + B) = t$$

which means that

$$5A = 1$$

and

$$2A + 5B = 0$$

after equating the coefficients of each power of t on each side of the equation. This gives $A = \frac{1}{5}$ and $B = -\frac{2}{25}$.
So

$$x = x_c + x_p = C_1 e^{-t} \cos(2t) + C_2 e^{-t} \sin(2t) + \frac{t}{5} - \frac{2}{25}.$$

Now we can determine the constants C_1 and C_2 from the initial conditions.

$$x(0) = C_1 - \frac{2}{25} = 0$$

and since

$$x' = e^{-t}\left(-C_1 \cos(2t) - C_2 \sin(2t) - 2C_1 \sin(2t) + 2C_2 \cos(2t)\right) + \frac{1}{5}$$

$$x'(0) = -C_1 + 2C_2 + \frac{1}{5} = 0$$

we find $C_1 = \frac{2}{25}$ and $C_2 = -\frac{3}{50}$.
So

$$x = \frac{2}{25}e^{-t}\cos(2\,t) - \frac{3}{50}e^{-t}\sin(2\,t) + \frac{t}{5} - \frac{2}{25}$$

and

$$x(2) = \frac{2}{25}e^{-2}\cos(4) - \frac{3}{50}e^{-2}\sin(4) + \frac{2}{5} - \frac{2}{25} \approx 0.31906844123817\ldots$$

(b) Approximate the solution $x(2)$ using two steps of the Improved Euler method. (Optional but recommended: compare this with one step of the Runge-Kutta fourth order method.)

Solution: If we introduce a new variable $v = x'$, then the second order ODE is equivalent to the system

$$x' = v$$
$$v' = -2v - 5x + t$$

We can think of the right-hand side of this as a vector-valued function

$$f(x, v, t) = \begin{pmatrix} v \\ -2v - 5x + t \end{pmatrix}$$

We start at $(x_0, v_0) = (0, 0)$ and $t_0 = 0$, with a stepsize of $(2-0)/2 = 1$. For the first step,

$$k_1 = f(0, 0, 0) = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

and

$$k_2 = f(0, 0, 1) = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

So

$$\begin{pmatrix} x_1 \\ v_1 \end{pmatrix} = \begin{pmatrix} x_0 \\ v_0 \end{pmatrix} + \frac{h}{2}(k_1 + k_2) = \begin{pmatrix} 0 \\ 0 \end{pmatrix} + \frac{1}{2}\left(\begin{pmatrix} 0 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 1/2 \end{pmatrix}\right) = \begin{pmatrix} 0 \\ 1/2 \end{pmatrix}$$

The second step has $k_1 = f(0, 1/2, 1) = \begin{pmatrix} 1/2 \\ 0 \end{pmatrix}$ and $k_2 = f(1/2, 1/2, 2) = \begin{pmatrix} 1/2 \\ -3/2 \end{pmatrix}$, with the result that the approximation is

$$x_2 = 0.5$$

Somewhat surprisingly, one step of the fourth-order Runge-Kutta method gives the approximation $x_1 = 0$, which is worse than the previous estimate. Two steps of Runge-Kutta gives the much better result $x_2 = \frac{9}{32} = 0.28125$.