Math 3280 Assignment 12, due Friday, December 7th.
This assignment covers material from chapters 7 and 10.

(1) Compute the Laplace transform of the function

$$v(t) = \begin{cases} 1 \text{ for } t \in [0,1] \\ 0 \text{ for } t \notin [0,1] \end{cases}$$

directly from the definition $\mathcal{L}(v) = \int_0^\infty e^{-st} v(t) dt$.

(2) Use the Laplace transform method to solve the initial value problem $x'' - x' - 2x = 0$, $x(0) = 0$, $x'(0) = 1$.

(3) Suppose two 50 liter tanks are connected by two pumps which transfer 10 liters/minute of fluid from each tank to the other. Suppose that the first tank initially contains 50 liters of brine at a concentration of 0.2 kg of salt per liter, and the other tank contains 50 liters of pure water.
   (a) Find the amount of salt in each tank as a function of time (you can assume that the tanks are well-stirred).
   (b) How long will it take for the amount of salt in the second tank to be within 1% of the amount of salt in the first tank?

(4) Find the general solution of $x' = Ax$ if

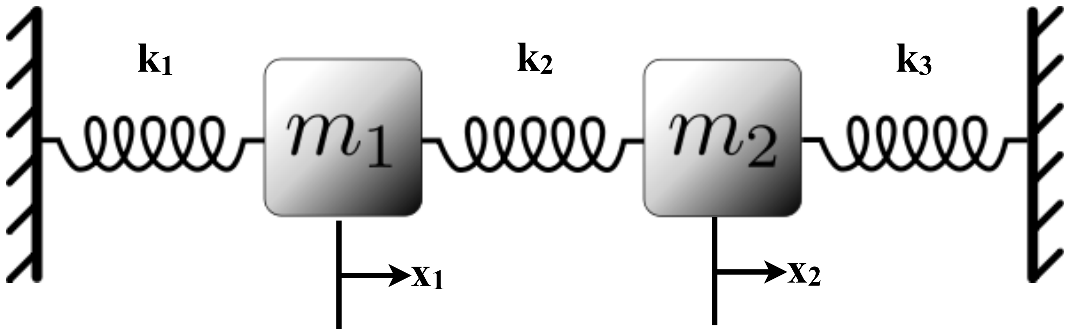
$$A = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 2 & 2 & 0 & 0 \\ 0 & 3 & 3 & 0 \\ 0 & 0 & 4 & 4 \end{pmatrix}$$

For the next three problems, consider two blocks of mass m_1 and m_2 connected by springs to each other and to walls as shown below. The displacement of the masses from their equilibrium positions are denoted by x_1 and x_2. The stiffness of the three springs are k_1, k_2, and k_3 as shown. Compute the natural frequencies and describe the natural modes of oscillation in each of the three following cases:

(5) $k_1 = k_2 = 4$ and $k_3 = 2$, and $m_1 = 2$, $m_2 = 1$.

(6) $k_1 = k_3 = 1$ and $k_2 = 4$, and $m_1 = m_2 = 1$.

(7) $k_1 = k_3 = 0$ and $k_2 = 4$, and $m_1 = m_2 = 1$.



(8) Find the error between the exact values of $x_1(1)$ and $x_2(1)$ and an approximation using Euler's method for the initial value problem $x_1' = 9x_1 + 5x_2$, $x_2' = -6x_1 - 2x_2$, $x_1(0) = 0$, $x_2(0) = 1$.

(9) Suppose a rocket is fired and after its initial burn it is at an altitude of $h = 5$ km with a velocity of 4 km/s straight up. Since it may reach a significant altitude, it could be too inaccurate to use a constant gravitational force. We will use the Newtonian law of gravity:

$$\frac{d^2h}{dt^2} = -\frac{gR^2}{(h+R)^2}$$

where h is the height above sea level in kilometers, $R = 6378$ km, and $g = 0.0098$ km/s^2.

Convert this to a system of first-order equations and use a numerical method to find the maximum height of the rocket. Your answer should have two digits of accuracy (within the assumptions of the model). Use of computers is highly encouraged for this problem.