Math 3280 Worksheet 29: Review problem solutions

Please let me know if you think you have found an error in these solutions.

(1) Find a basis for the subspace of solutions to the linear system

$$\begin{aligned} 2y + z &= 0 \\ x + 6y - z &= 0 \end{aligned}$$

Solution: The main step is to row reduce the coefficient matrix:

$$\begin{pmatrix} 0 & 2 & 1 \\ 1 & 6 & -1 \end{pmatrix} \xrightarrow{\textit{Swap } R_1,R_2} \begin{pmatrix} 1 & 6 & -1 \\ 0 & 2 & 1 \end{pmatrix} \xrightarrow{-3R_2+R_1} \begin{pmatrix} 1 & 0 & -4 \\ 0 & 2 & 1 \end{pmatrix}$$

$$\xrightarrow{R_2/2} \begin{pmatrix} 1 & 0 & -4 \\ 0 & 1 & 1/2 \end{pmatrix}$$

Now for any solution to the system we can write the pivot variables x and y in terms of the free variable z:

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 4z \\ -z/2 \\ z \end{pmatrix} = z \begin{pmatrix} 4 \\ -1/2 \\ 1 \end{pmatrix}$$

which implies that $\{ \begin{pmatrix} 4 \\ -1/2 \\ 1 \end{pmatrix} \}$ is a basis for the solution space.

(2) Find the general solution to $y^{(4)} + 6y''' + 13y'' = 0$.

Solution: The characteristic equation factors:

$$r^4 + 6r^3 + 13r^2 = r^2(r - (-3 + 2i))(r - (-3 + 2i)) = 0$$

(the complex roots could be found using the quadratic equation after factoring out the r^2).

So there is a double root at 0 and complex conjugate roots $-3 \pm 2i$. This means the general solution is

$$y = C_1 + C_2 t + C_3 e^{-3t}\cos(2t) + C_4 e^{-3t}\sin(2t).$$

(3) Solve the initial value problem $y'' + 2y' = 3 + 4\sin(2t)$, $y(0) = 0$, $y'(0) = 2$.

Solution: The characteristic equation is $r^2 + 2r = r(r + 2) = 0$, with roots 0 and -2. So the homogeneous solutions is $y_c = C_1 + C_2 e^{-2t}$.

Normally we would choose a particular solution of the form $A + B\sin(2t) + C\cos(2t)$, but since the constant A is contained in the homogeneous solution we multiply it by t to get $y_p = A\sin(2t) + B\cos(2t) + Ct$.

Another way to obtain this form for the particular solution y_p is to use the method of annihilation. Since the constant 3 is annihilated by the operator $\frac{d}{dt}$, and $4\sin(2t)$ is annihilated by $(\frac{d^2}{dt^2} + 1)$, if we apply these to the original differential equation we get:

$$(\frac{d}{dt})(\frac{d^2}{dt^2} + 1)(\frac{d}{dt} + 2)(\frac{d}{dt})y = (\frac{d}{dt})(\frac{d^2}{dt^2} + 1)(3 + 4\sin(2t)) = 0$$

Any solution of the original ODE satisfies this ODE as well. It is homogeneous with characteristic polynomial $(r^2 + 1)r^2(r + 2)$. This tells us that the particular solution must be of the form $A\sin(2t) + B\cos(2t) + Ct + D + Ee^{-2t}$. We do not need to include the last two terms since they are part of the homogeneous solution to the original ODE.

Now we compute

$$y_p'' + 2y_p' = (-4A - 4B)\sin(2t) + (4A - 4B)\cos(2t) + 2C = 3 + 4\sin(2t)$$

so $2C = 3$, $-4A - 4B = 4$, and $4A - 4B = 0$. These are solved (by row-reduction or substitution or inspection) to get $A = B = -1/2$, and $C = 3/2$.

So the solution is

$$y = y_c + y_p = C_1 + C_2e^{-2t} - \sin(2t)/2 - \cos(2t)/2 + 3t/2.$$

Plugging in the initial conditions to y and y' at $t = 0$ gives us the equations

$$C_1 + C_2 - 1/2 = 0, \qquad -2C_2 + 1/2 = 2.$$

So $C_2 = -3/4$ and $C_1 = 5/4$.

The solution to the initial value problem is therefore

$$y = 5/4 - 3/4e^{-2t} - \sin(2t)/2 - \cos(2t)/2 + 3t/2$$

(4) Use the method of variation of parameters to find the general solution of $y'' + 4y' + 4y = t^{-2}e^{-2t}$.

First we solve the associated homogeneous problem, which has characteristic polynomial $r^2 + 4r + 4 = (r + 2)^2$. The double root at -2 means that $y_c = C_1e^{-2t} + C_2te^{-2t}$. We can use $y_1 = e^{-2t}$ and $y_2 = te^{-2t}$ as a basis for the homogeneous solution space.

The Wronskian of y_1 and y_2 is

$$W = \det\begin{pmatrix} e^{-2t} & te^{-2t} \\ -2e^{-2t} & e^{-2t} - 2te^{-2t} \end{pmatrix} = e^{-2t}\det\begin{pmatrix} 1 & te^{-2t} \\ -2 & e^{-2t} - 2te^{-2t} \end{pmatrix}$$

$$= e^{-4t} \det \begin{pmatrix} 1 & t \\ -2 & 1-2t \end{pmatrix} = e^{-4t} - 2t + 2t = e^{-4t}$$

The particular solution is $y_p = u_1 y_1 + u_2 y_2$; with $f(t) = t^{-2} e^{-2t}$ we have

$$u_1 = -\int \frac{y_2 f}{W}\, dt = -\int 1/t\, dt = -\ln(t)$$
$$u_2 = \int \frac{y_1 f}{W}\, dt = \int 1/t^2\, dt = -1/t$$

and $y_p = u_1 y_1 + u_2 y_2 = -\ln(t) e^{-2t} - e^{-2t}$.

Since $-e^{-2t}$ is a part of the homogeneous solution, we do not need to include it in y_p.

So the general solution is

$$y = C_1 e^{-2t} + C_2 t e^{-2t} - \ln(t) e^{-2t}.$$