Math 3280 Practice Final

This is longer than the actual exam, which will be 8 to 10 questions (some might be multiple choice). You are allowed up to two sheets of notes (both sides) and a calculator, although any use of a calculator must be indicated. On numerical method probems (e.g. Euler's method) the use of a (non-internet capable) calculator is expected.

(1) Find the general solution to $(1+t)y' + y = \cos t$.

(2) Rewrite the initial value problem $y''' + y'' + y = t$, $y(0) = y'(0) = y''(0) = 0$ as an equivalent first-order system.

(3) Find the general solution to the system

$$\frac{d}{dt}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 2 & 4 \\ -1 & -3 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}.$$

(4) Are the vectors $v_1 = (1, 2, 3, 4)$, $v_2 = (2, -2, 4, 2)$, and $v_3 = (0, -3, -1, -3)$ linearly independent? If not, write one of them as a linear combination of the other two.

(5) Solve the initial value problem $y'' + y = \cos x$, $y'(0) = 0$, $y(0) = -\frac{1}{2}$.

(6) Use Euler's, the Improved Euler's, or the Runge-Kutta method to numerically approximate $y(2)$ to two digits of accuracy if $y' = t + \sqrt{y}$ and $y(0) = 1$.

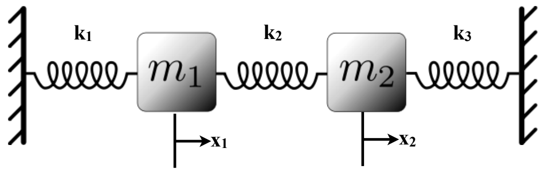
(7) Find the general solution to the system

$$\frac{d}{dt}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 & -5 \\ 1 & 3 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}.$$

(8) Find the Laplace transform $X(s) = \mathcal{L}(x(t))$ if $x'' + 8x' + 15x = 0$ and $x(0) = 0$, $x'(0) = 1$. Then find the solution $x(t)$.

(9) What is the **form** of the general solution to the ODE $y''' - 4y'' + 14y' - 20y = te^t \cos(3t) + t^2$. Hint: one of the roots of the characteristic polynomial of the left-hand side is 2. For extra credit find the values of the constants in the particular solution.

(10) Consider a mass-spring system with two masses of mass m_1 and m_2. Mass 1 is connected to a wall with a spring of stiffness k_1 and to mass 2 with a spring of stiffness k_2. Mass 2 is a connected to a second wall with a spring of stiffness k_3, as shown below. Their displacements from the equilibrium are x_1 and x_2, which we will combine into a vector $x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$. Then if $x'' = Ax$, show that the real parts of the eigenvalues of A must be negative if the masses and spring constants are positive.



(11) Use either the Laplace transform method or the eigenvalue/eigenvector method to find the steady state solution to the initial value problem $x' = -x - z$, $y' = -x - y$, $z' = 2x + z$, $x(0) = 0$, $y(0) = 0$, $z(0) = 2$.

(12) Find the equilibria of the system $x' = 2y^3 - 2x$, $y' = x^2 - 1$, and determine their stability by computing the eigenvalues of the linearized systems.

(13) Three identical, well-stirred tanks of with 100 liters of water in each tank are connected in series with tank 1 pumping 10 liter/minute into tank 2, tank 2 pumping 10 liter/minute into tank 3, and tank 3 pumping 10 liter/minute into tank 1. If tank 1 initially has 500 grams of salt dissolved in it, and the other two tanks start at time $t = 0$ with no salt, which of the following initial value problems describes the amounts of salt in grams in each tank (x_1 = salt in tank 1, x_2 = salt in tank 2, x_3 = salt in tank 3).

(a) $x_1' = \frac{1}{10}x_3 - \frac{1}{10}x_1$ $\quad x_2' = \frac{1}{10}x_1 - \frac{1}{10}x_2$ $\quad x_3' = \frac{1}{10}x_2 - \frac{1}{10}x_3$

$x_1(0) = 0$, $x_2(0) = 0$, $x_3(0) = 0$.

(b) $x_1' = \frac{1}{100}x_2 - \frac{1}{10}x_1$ $\quad x_2' = \frac{1}{100}x_3 - \frac{1}{10}x_2$ $\quad x_3' = \frac{1}{100}x_1 - \frac{1}{10}x_3$

$x_1(0) = 500$, $x_2(0) = 0$, $x_3(0) = 0$.

(c) $x_1' = \frac{1}{10}x_3 - \frac{1}{10}x_1 \qquad x_2' = \frac{1}{10}x_1 - \frac{1}{10}x_2 \qquad x_3' = \frac{1}{10}x_2 - \frac{1}{10}x_3$
$x_1(0) = 500,\ x_2(0) = 0,\ x_3(0) = 0.$

(d) $x_1' = \frac{1}{10}x_3 + \frac{1}{10}x_1 \qquad x_2' = \frac{1}{10}x_1 + \frac{1}{10}x_2 \qquad x_3' = \frac{1}{10}x_2 + \frac{1}{10}x_3$
$x_1(0) = 500,\ x_2(0) = 0,\ x_3(0) = 0.$

(14) What is the dimension of the field of complex numbers when it is considered as a vector space over the field of real numbers? Justify your answer by finding a basis.