Math 3280 Practice Midterm 1 Solutions

(1) Solve the initial value problem $y' = xy^3$, $y(0) = 2$.

Solution: This is seperable, so we divide by y^3 and integrate to get

$$\int y^{-3} = \int x$$

$$-y^{-2}/2 = x^2/2 + C.$$

Solving for y, we get $y = \pm 1/\sqrt{-x^2 - 2C}$. We choose the positive root now, because the initial condition has a positive y-value. Using the initial condition we can find $y(0) = 2 = 1/\sqrt{-2C}$, so $C = -1/8$ and we can simplify the answer to $y = 2/\sqrt{1 - 4x^2}$.

(2) Find the general solution to the ODE $xy' = 2y + x^3/(1 + x^2)$.

Solution: This is a linear ODE. First put it in standard form

$$y' + (-2/x)y = x^2/(1 + x^2).$$

So $P = -2/x$ and $Q = x^2/(1 + x^2)$. The integrating factor is

$$\rho = e^{\int -2/x dx} = e^{\ln(x^{-2})} = x^{-2}$$

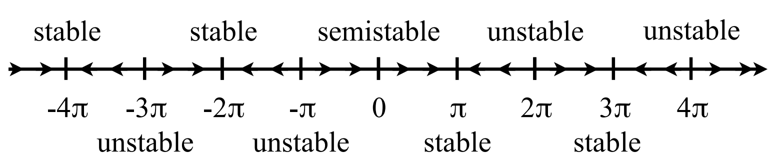
Then

$$\rho y = \int \rho Q dx = \arctan(x) + C$$

and $y = x^2 \arctan(x) + Cx^2$.

(3) Draw a phase diagram showing the equilibria, their stability, and the direction of the solutions for the ODE $y' = y^3 \sin y$. Justify your answers.

Solution: the function $y^3 \sin y$ has zeros at $n\pi$ with $n \in \mathbb{Z}$. The derivative of this function (with respect to y) is $y^3 \cos y + 3y^2 \sin y$; at the equilibria this is equal to $(n\pi)^3$ when n is even and $-(n\pi)^3$ when n is odd. So apart from $y = 0$ the stability of the equilibria alternates, with even positive multiples of π being unstable equilibria and the uneven positive multiples stable. For negative n this is reversed with the uneven multiples being unstable. Near $y = 0$, the solutions always increase so this equilibria is semistable.



(4) Suppose an object moves through a fluid with resistance proportional to its velocity $v = \frac{dx}{dt}$, so $\frac{dv}{dt} = -kv$. If the initial velocity is $v(0) = 2$ meters/second, and the initial position is $x(0) = 0$, find $x(t)$ (first find $v(t)$).

Solution: see problem 2.3 #2. The solution is $x = 2(1 - e^{-kt})/k$.

(5) Your friend Don asks for your help in a homicide investigation. The corpse of the victim was found at midnight with a body temperature of $T(0) = T_0 = 30°C$ in a room with a constant ambient temperature of $A = 25°C$. Six minutes later, the body's temperature had fallen to $T(.1) = 29.9°C$ (using time t in hours). If a normal body temperature is $37°C$, when did the victim die? (Assume that Newton's law of cooling holds: the temperature changes at a rate proportional to the difference between the ambient temperature and the body temperature. Also, use the information about the change in temperature to approximate $\frac{dT}{dt}(0)$.)

Solution: We approximate T' by the difference quotient $\frac{\Delta T}{\Delta t} = \frac{30-29.9}{0-.1} = -1$, so assume $T'(0) = -1$ (in degrees Celsius/hour). Then we can solve for the proportionality constant k from $-1 = k(25-30)$, so $k = 1/5$. This ODE is seperable with solution $T(t) = A + (T_0 - A)e^{-kt} = 25 + 5e^{-t/5}$ (see p.39 in the text for an example). Now we solve for t in the equation $37 = 25 + 5e^{-t/5}$, or $e^{-t/5} = 12/5$. After taking the logarithm and cleaning up we find the solution is $t = -5\ln(12/5) \approx -4.38$ hours. So you estimate the time as $7:37$ pm the previous evening.

(6) Use the improved Euler's method with one, two, and four steps to approximate the $y(2)$ if $y(1) = 1$ and $\frac{dy}{dx} = x/y + 3y/x$. How many digits would you trust in your last answer? For extra credit, find an answer correct to at least five digits.

Solution: For the one step case, the stepsize is 1 and the two slopes are $f(1,1) = 4$ and $f(1+1, 1+4) = 7.9$. So the improved Euler approximation is $1 + (4+7.9)/2 = 139/20 = 6.95$.

In two steps we find that $y_1 = 29/8$ and $y_2 = \frac{6746329}{802720} \approx 8.4$. You would not be expected to keep track of the exact form.

The four step case has intermediate y values $y_1 \approx 2.18$, $y_2 \approx 3.86$, $y_3 \approx 6.18$, and finally $y_4 \approx 9.23$.

Note that the true solution is $\sqrt{94} \approx 9.7$ so we're not that close.

(7) Find the inverse of the matrix $A = \begin{bmatrix} a & 0 & 0 \\ 0 & -1 & 3 \\ 0 & -1 & -2 \end{bmatrix}$, where a is a nonzero real number.

Solution: $A^{-1} = \begin{pmatrix} \frac{1}{a} & 0 & 0 \\ 0 & -\frac{2}{5} & -\frac{3}{5} \\ 0 & \frac{1}{5} & -\frac{1}{5} \end{pmatrix}$

(8) Compute the determinant of

$$A = \begin{pmatrix} 1 & 2 & 2 & -2 \\ 1 & 4 & 5 & -6 \\ 0 & 2 & 4 & -3 \\ -1 & -6 & -8 & 9 \end{pmatrix}.$$

Solution: This is easier after a little row reduction - adding the first row to the last, and subtracting the first row from the second. Then we can expand using the first column, and for the three by three minor we expand using the top row. There are many other ways of doing this though.

$$det(A) = \begin{vmatrix} 1 & 2 & 2 & -2 \\ 0 & 2 & 3 & -4 \\ 0 & 2 & 4 & -3 \\ 0 & -4 & -6 & 7 \end{vmatrix} = 1 \begin{vmatrix} 2 & 3 & -4 \\ 2 & 4 & -3 \\ -4 & -6 & 7 \end{vmatrix}$$

$$= 2 \begin{vmatrix} 4 & -3 \\ -6 & 7 \end{vmatrix} - 3 \begin{vmatrix} 2 & -3 \\ -4 & 7 \end{vmatrix} + (-4) \begin{vmatrix} 2 & 4 \\ -4 & -6 \end{vmatrix}.$$

$$= 2(28 - 18) - 3(14 - 12) - 4(-12 + 16) = -2.$$

(9) Express $w = (7, -6, 14, 0)$ as a linear combination of $v_1 = (2, 3, 4, 0)$ and $v_2 = (-1, 4, -2, 0)$, or show that it is impossible to do so.

Solution: We are looking for numbers c_1 and c_2 such that $c_1 v_1 + c_2 v_2 = w$. This can be written as the matrix equation

$$\begin{pmatrix} 2 & -1 \\ 3 & 4 \\ 4 & -2 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} 7 \\ -6 \\ 14 \\ 0 \end{pmatrix}$$

The augmented coefficient matrix for this system is

$$\begin{pmatrix} 2 & -1 & 7 \\ 3 & 4 & -6 \\ 4 & -2 & 14 \\ 0 & 0 & 0 \end{pmatrix}$$

which can be row-reduced to

$$\begin{pmatrix} 1 & 0 & 2 \\ 0 & 1 & -3 \end{pmatrix}$$

(zero rows have been removed).

So it is possible, with $c_1 = 2$ and $c_2 = -3$.

(10) Find a basis for the subspace defined by the following equations for $(x_1, x_2, x_3, x_4) \in \mathbb{R}^4$:

$$-3x_1 - 3x_2 + 2x_3 - 2x_4 = 0$$
$$x_1 - 3x_2 - 4x_3 = 0$$
$$7x_1 + 15x_2 + 2x_3 + 6x_4 = 0$$

Solution: We need to row-reduce the coefficient matrix:

$$\begin{pmatrix} -3 & -3 & 2 & -2 \\ 1 & -3 & -4 & 0 \\ 7 & 15 & 2 & 6 \end{pmatrix} \xrightarrow{Swap\ R_1,R_2} \begin{pmatrix} 1 & -3 & -4 & 0 \\ -3 & -3 & 2 & -2 \\ 7 & 15 & 2 & 6 \end{pmatrix}$$

$$\xrightarrow{3R_1+R_2,-7R_1+R_3} \begin{pmatrix} 1 & -3 & -4 & 0 \\ 0 & -12 & -10 & -2 \\ 0 & 36 & 30 & 6 \end{pmatrix} \xrightarrow{3R_2+R_3} \begin{pmatrix} 1 & -3 & -4 & 0 \\ 0 & -12 & -10 & -2 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\xrightarrow{R_2/(-12)} \begin{pmatrix} 1 & -3 & -4 & 0 \\ 0 & 1 & 5/6 & 1/6 \\ 0 & 0 & 0 & 0 \end{pmatrix} \xrightarrow{3R_2+R_1} \begin{pmatrix} 1 & 0 & -3/2 & 1/2 \\ 0 & 1 & 5/6 & 1/6 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

This tells us that the first two variables are basic, or pivot, variables, while x_3 and x_4 are free variables. The number of free variables = the dimension of the solution space = the number of elements in any basis. We can read a basis off the coefficients of the free variables:

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 3x_3/2 - x_4/2 \\ -5x_3/6 - x_4/6 \\ x_3 \\ x_4 \end{pmatrix} = x_3 \begin{pmatrix} 3/2 \\ -5/6 \\ 1 \\ 0 \end{pmatrix} + x_4 \begin{pmatrix} -1/2 \\ -1/6 \\ 0 \\ 1 \end{pmatrix}$$

I.e., one basis is $\left\{ \begin{pmatrix} 3/2 \\ -5/6 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -1/2 \\ -1/6 \\ 0 \\ 1 \end{pmatrix} \right\}$.

(11) Match each of the following six differential equations with the corresponding slope field.

Solution:

(a) $\dfrac{dy}{dx} = \sin(x)$ matches to IV.

(b) $\dfrac{dy}{dx} = \dfrac{y}{(1+x^2)}$ matches to V.

(c) $\dfrac{dy}{dx} = 2y$ matches to I.

(d) $\dfrac{dy}{dx} = 1$ matches to VI.

(e) $\dfrac{dy}{dx} = \dfrac{x}{(1+y^2)}$ matches to III.

(f) $\dfrac{dy}{dx} = \sin(y)$ matches to II.