Math 3280 Review Worksheet Solutions

(1) Use the Laplace transform to solve the initial value problem $y'' + y = \cos(t)$, $y(0) = 0$, $y'(0) = 1$.

Solution:
The transform of the ODE is

$$s^2 Y - 1 + Y = \frac{s}{s^2+1}$$

Solving for $\mathcal{L}(y) = Y$ we get

$$Y = \frac{1}{s^2+1} + \frac{s}{(s^2+1)^2}$$

Normally we would combine everything on the right hand side and then use partial fraction decomposition, but in this case we already have functions we can invert using the transforms $\mathcal{L}(\sin(t)) = \frac{1}{s^2+1}$ and

$$\mathcal{L}(t\sin(t)) = -\frac{d}{ds}\mathcal{L}(\sin(t)) = \frac{2s}{(s^2+1)^2}$$

so $y = \sin(t) + t\sin(t)/2$.

(2) Solve the same initial value problem using undetermined coefficients (i.e. use the decomposition $y = y_h + y_p$).

Solution: First we compute the homogeneous solution $y_c'' + y_c = 0$. The characteristic equation is $r^2 + 1 = 0$, with roots $\pm i$. This means that $y_c = C_1\cos(t) + C_2\sin(t)$.

Next we find the particular solution. Usually we would use $A\cos(t) + B\sin(t)$, but this overlaps with the homogeneous solution so we multiply by t to get $y_p = At\cos(t) + Bt\sin(t)$.

To substitute y_p into the equation we need its second derivative:

$$y_p' = Bt\cos(t) - At\sin(t) + A\cos(t) + B\sin(t)$$
$$y_p'' = -At\cos(t) - Bt\sin(t) + 2B\cos(t) - 2A\sin(t)$$

So $y_p'' + y_p = 2B\cos(t) - 2A\sin(t) = \cos(t)$ and we see that $B = 1/2$ and $A = 0$. So

$$y = y_c + y_p = C_1\cos(t) + C_2\sin(t) + t\sin(t)/2$$

Now we can use the initial conditions to find that $C_1 = 0$ and $C_2 + 1/2 = 1$, so $C_2 = 1/2$. Finally: $y = \sin(t) + t\sin(t)/2$, as above.