Math 3280 Worksheet 45 Solutions

(1) Consider three tanks, each containing 1 liter. r liters of fluid per minute is sent from tank 1 to 2, from tank 2 to tank 1, from tank 2 to tank 3, and from tank 3 to tank 2. Initially tank 1 is filled with pure water, tank 2 is filled with brine at a concentration of 2 grams of salt per liter, and tank 3 is filled with brine at a concentration of 4 grams per liter.

(a) Write down the differential equations and initial conditions for the amount of salt in each tank (x_1, x_2, and x_3).

Solution:

$$x_1' = -rx_1 + rx_2$$
$$x_2' = rx_1 - 2rx_1 + rx_3$$
$$x_3' = rx_2 - rx_3$$

$$x_1(0) = 0$$
$$x_2(0) = 2$$
$$x_3(0) = 4$$

(b) Solve the differential equations.

Solution:

$$x' = r\begin{pmatrix} -1 & 1 & 0 \\ 1 & -2 & 1 \\ 0 & 1 & -1 \end{pmatrix}$$

The eigenvalues of the coefficient matrix are $0, -r, -3r$, with eigenvectors $\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$, $\begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$, $\begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}$ respectively.

So

$$x = C_1\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} + C_2e^{-rt}\begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} + C_3e^{-3rt}\begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}$$

The initial conditions give us a linear system whose solution is $C_1 = 2$, $C_2 = 2$, $C_3 = 0$. So

$$x = 2\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} + 2e^{-rt}\begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$$

(c) Find a value of r so that the concentration of salt in tank 3 is three times that of tank 1 at time $t = 1$ minute.

Solution:
Substituting $t = 1$ into the equation $x_3/x_1 = 3$ we can solve for $r = \ln(2)$.

(2) The spread of many diseases are modeled by various SIR ODE models, where SIR is an acronym for Susceptible, Infected, and Recovered. In the following version, we assume a population has a constant proportional death rate of d and a birth rate of b. The disease is transmitted at a rate cIS, and infected people recover at a rate proportional to I, giving the equations:

$$\frac{dS}{dt} = b - dS - cIS$$
$$\frac{dI}{dt} = cIS - (d + g)I$$
$$\frac{dR}{dt} = gI - dR$$

For a population with $b = d = 1$, when is the disease-free equilibrium point (disease free meaning $I = R = 0$) stable?

Solution:
If $I = R = 0$, then if $\frac{dS}{dt} = 0$ we must have $S = b/d$ which is 1 for $b = d = 1$. So the equilibrium point is $(1, 0, 0)$.
The Jacobian is

$$J = \begin{pmatrix} -1 & -cS & 0 \\ cI & cS - g - 1 & 0 \\ 0 & g & -d \end{pmatrix} |_{S=1,I=0,R=0} = \begin{pmatrix} -1 - cI & -c & 0 \\ 0 & c - g - 1 & 0 \\ 0 & g & -d \end{pmatrix}$$

Expanding the determinant along the first or last column we find

$$det(J - \lambda I) = (-\lambda - 1)^2(c - g - 1 - \lambda)$$

so the eigenvalues are $-1, -1, c - g - 1$. In order for the equilibrium point to be stable, we need all of the real parts of the eigenvalues to be nonpositive, so $c - g - 1 \leq 0$, or $1 + c \geq g$. This quantifies the fact that the recovery rate from infection, g, must be one unit larger than the transmission interaction rate c.