

- (1) Sherlock Holmes is awoken by a phone call from a policeman at 3:30am. A body has been discovered and foul play is suspected. Sherlock tells the police to determine the temperature of the body and, when he arrives at the scene 45 minutes later, he takes the temperature again. The two readings that cold 60 degree F morning were 80 degrees F and 70 degrees F.

- (a) Using Newton's Law of Cooling ($T' = -k(T - A)$), draw a phase diagram for T labeled with the equilibria, and arrows indicating the direction of the solutions.



- (b) When was the latest time that the body was 98.6? (Note that in this problem, the readings are taken far enough apart that you should not use a secant slope estimate to determine the parameter in the ODE.)

Solution: Newton's law of cooling has the solution

$$T(t) = A + (T_0 - A)e^{-k(t-t_0)}$$

if $T(t_0) = T_0$.

If we choose to consider $t_0 = 0$ at 3:30 am, and use t in minutes, then $T_0 = 80$ and we can first determine k from the condition

$$T(45) = 70 = 60 + (80 - 60)e^{-45k}$$

After solving for $k = -\ln(.5)/45 \approx 0.0154$, we can solve for the value of t which gives $98.6 = 60 + 20e^{-kt}$. This gives $t = -42.7$ so the time of death was about 2:47 am.

- (2) The model $y' = -ky$ is a little too simple to provide a good model for some pharmacokinetic phenomena, such as the concentration of alcohol in the blood after having a drink. To be more accurate we can use a two-tank model.

Let tank 1 model the stomach, with a volume of 1 liter and an initial concentration of 2% alcohol. Assume fluid is absorbed into the bloodstream (tank 2) from the stomach at a rate of 100 milliliters per minute, and that the volume of the stomach stays constant (i.e. replaced without additional alcohol). The volume of the blood is 5 liters, initially with no alcohol, and fluid is exchanged between the blood and the body tissues at a rate of 100 milliliters per minute. You can assume that all alcohol transferred into the body is metabolized, so none of it returns to the bloodstream. Find the maximum concentration of alcohol in the bloodstream.

Solution: Let x_1 and x_2 denote the amounts of alcohol in the two compartments respectively.

Then our equations are:

$$x_1' = -x_1/10, \quad x_1(0) = 0.02$$

$$x_2' = x_1/10 - x_2/50, \quad x_2(0) = 0$$

The first equation can be solved (by reference, or using its linearity, or its separability) to obtain $x_1 = 0.02e^{-t/10}$.

This can be substituted into the differential equation for x_2 , and then re-arranged into standard form:

$$x_2' + x_2/50 = 0.002e^{-t/10}$$

We will use the linear solution

$$x_2 = \frac{C}{\rho} + \frac{1}{\rho} \int \rho Q dt$$

The integrating factor is $\rho = e^{t/50}$. After integrating

$$\int \rho Q dt = 0.002 \int e^{t/50} e^{-t/10} dt = -0.025e^{-0.08t}$$

and using the initial condition to determine the constant, we obtain the solution for x_2 :

$$x_2 = 0.025(e^{-0.02t} - e^{-t/10})$$

To find the maximum we differentiate x_2 , set the derivative equal to zero, and solve for t :

$$t = \frac{25 \ln(5)}{2} \approx 20.12 \text{ minutes}$$

Then we can evaluate

$$x_2(20.12) \approx 0.013$$

So the maximum concentration in the blood is $0.01337/5 = 0.267\%$.