Worksheet 45 and 46 solutions

(1) Compute the inverse of

$$A = \begin{bmatrix} \cos(\theta) & -\sin(\theta) & 0 \\ \sin(\theta) & \cos(\theta) & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

Solution:

$$A^{-1} = \begin{bmatrix} \cos(\theta) & \sin(\theta) & 0 \\ -\sin(\theta) & \cos(\theta) & 0 \\ 0 & 0 & 1/2 \end{bmatrix}$$

(2) Are the vectors $v_1 = \begin{pmatrix} 1 \\ -1 \\ 0 \\ 0 \end{pmatrix}$, $v_2 = \begin{pmatrix} 0 \\ 1 \\ -1 \\ 0 \end{pmatrix}$, $v_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \\ -1 \end{pmatrix}$, $v_4 = \begin{pmatrix} -1 \\ 0 \\ 0 \\ 1 \end{pmatrix}$ linearly independent? If not, write one of them as a linear combination of the others.

Solution:
The condition that $c_1v_1 + c_2v_2 + c_3v_3 + c_4v_4 = 0$ can be written as a matrix-vector system

$$\begin{pmatrix} 1 & 0 & 0 & -1 \\ -1 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & -1 & 1 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \end{pmatrix} 0$$

The reduced row echelon form of the coefficient matrix is

$$\begin{pmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

which has only 3 pivots - the last column corresponds to the coefficient c_4, which is a free variable. So the vectors are not independent. We can choose $c_4 = 1$, which then implies $c_1 = c_2 = c_3 = 1$, and there is the linear relation

$$v_1 + v_2 + v_3 + v_4 = 1$$

which can be solved for v_4, for example, to get

$$v_4 = -v_1 - v_2 - v_3.$$

(3) A matrix P is an *orthogonal projection* if $P^2 = P$ and $P^T = P$. Find the 3×3 orthogonal projection P that projects any 3-D vector $\begin{bmatrix} x \\ y \\ z \end{bmatrix}$ onto the line spanned by $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$.

Hint: for this projection, $P \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ and $Pb = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$ for any b that is perpendicular to $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$, such as $b = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$.

Solution:

The condition that $Pv = 0$ for vectors v perpendicular to $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ implies that all of the entries of P are equal. Then the condition $P \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ implies that each entry is 1/3, so

$$P = \frac{1}{3} \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}$$

(4) Consider an long cascade of tanks, each containing 1 liter of water. Each tank drains into the next at a rate of 1 liter per hour. Initially the first tank contains 1 gram of salt dissolved into it, but it is being refilled with pure water at a rate of 1 liter per hour. The other tanks in the cascade are initially filled with pure water. Compute how much salt is in the nth tank at time t.

Solution: The amount of salt in the first tank, $x_1(t)$, has the initial condition $x_1(0) = 1$ and ODE $x_1' = -x_1$. The solution to this is $x_1 = e^{-t}$.

For the nth tank, $x_n' = -x_n + x_{n-1}$. If we know x_{n-1}, then this is a nonhomogeneous first order ODE. In standard form, $x_n' + x_n = x_{n-1}$.

We can show by induction that $x_n = \frac{t^n e^{-t}}{n!}$. The integrating factor for the ODE is e^t, so

$$x_n = Ce^{-t} + e^{-t} \int x_{n-1} e^t \, dt$$

Our inductive assumption is that $x_{n-1} = \frac{t^{n-1} e^{-t}}{(n-1)!}$,

$$x_n = Ce^{-t} + e^{-t} \int \frac{t^{n-1} e^{-t}}{(n-1)!} e^t \, dt = Ce^{-t} + e^{-t} \int \frac{t^{n-1}}{(n-1)!} \, dt = Ce^{-t} + e^{-t} \frac{t^n}{n!}$$

and since $x_n(0) = 0$, $C = 0$, so $x_n = \frac{t^n e^{-t}}{n!}$.

(5) The spread of many diseases are modeled by various SIR ODE models, where SIR is an acronym for Susceptible, Infected, and Recovered. In the following version, we assume a population has a constant proportional death rate of d and a birth rate of b. The disease is transmitted at a rate cIS, and infected people recover at a proportional rate I, giving the equations:

$$\frac{dS}{dt} = b - dS - cIS$$
$$\frac{dI}{dt} = cIS - (d+g)I$$
$$\frac{dR}{dt} = gI - dR$$

For a population with $b = d = 1$, when is the disease-free equilibrium point (disease free meaning $I = R = 0$) stable?

Solution:

If $I = R = 0$, then if $\frac{dS}{dt} = 0$ we must have $S = b/d$ which is 1 for $b = d = 1$. So the equilibrium point is $(1, 0, 0)$.

The Jacobian is

$$J = \begin{pmatrix} -1 & -cS & 0 \\ cI & cS - g - 1 & 0 \\ 0 & g & -d \end{pmatrix} |_{S=1,I=0,R=0} = \begin{pmatrix} -1 - cI & -c & 0 \\ 0 & c - g - 1 & 0 \\ 0 & g & -d \end{pmatrix}$$

Expanding the determinant along the first or last column we find

$$det(J - \lambda I) = (-\lambda - 1)^2 (c - g - 1 - \lambda)$$

so the eigenvalues are $-1, -1, c - g - 1$. In order for the equilibrium point to be stable, we need all of the real parts of the eigenvalues to be nonpositive, so $c - g - 1 \leq 0$, or $1 + c \geq g$. This quantifies the fact that the recovery rate from infection, g, must be (one unit) larger than the transmission interaction rate c for the disease to disappear.