

Math 3280 Practice Final

This is longer than the actual exam, which will be 7 or 8 questions (some might be multiple choice). You are allowed up to two sheets of notes (both sides) and a calculator, although any use of a calculator must be indicated.

- (1) Find the general solution to $(1 + t)y' + y = \cos t$.
- (2) Rewrite the initial value problem $y''' + y'' + y = t$, $y(0) = y'(0) = y''(0) = 0$ as an equivalent first-order system.
- (3) Find the general solution to the system

$$\frac{d}{dt} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 2 & 4 \\ -1 & -3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}.$$

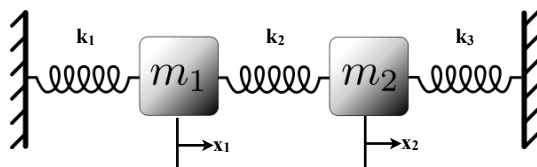
- (4) Are the vectors $v_1 = (1, 2, 3, 4)$, $v_2 = (2, -2, 4, 2)$, and $v_3 = (0, -3, -1, -3)$ linearly independent? If not, write one of them as a linear combination of the other two.
- (5) Solve the initial value problem $y'' + y = \cos x$, $y'(0) = 0$, $y(0) = -\frac{1}{2}$.
- (6) Use Euler's, the Improved Euler's, or the Runge-Kutta method to numerically approximate $y(2)$ to two digits of accuracy if $y' = t + \sqrt{y}$ and $y(0) = 1$.

- (7) Find the general solution to the system

$$\frac{d}{dt} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 & -5 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}.$$

- (8) Find the Laplace transform $X(s) = \mathcal{L}(x(t))$ if $x'' + 8x' + 15x = 0$ and $x(0) = 0$, $x'(0) = 1$. Then find the solution $x(t)$.
- (9) What is the **form** of the general solution to the ODE $y''' - 4y'' + 14y' - 20y = te^t \cos(3t) + t^2$. Hint: one of the roots of the characteristic polynomial of the left-hand side is 2. For extra credit find the values of the constants in the particular solution.

- (10) Consider a mass-spring system with two masses of mass m_1 and m_2 . Mass 1 is connected to a wall with a spring of stiffness k_1 and to mass 2 with a spring of stiffness k_2 . Mass 2 is connected to a second wall with a spring of stiffness k_3 , as shown below. Their displacements from the equilibrium are x_1 and x_2 , which we will combine into a vector $x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$. Then if $x'' = Ax$, show that the eigenvalues of A must be negative if the masses and spring constants are positive.



- (11) Use either the Laplace transform method or the eigenvalue/eigenvector method to find the steady state solution to the initial value problem $x' = -x - z$, $y' = -x - y$, $z' = 2x + z$, $x(0) = 0$, $y(0) = 0$, $z(0) = 2$.
- (12) Find the equilibria of the system $x' = 2y^3 - 2x$, $y' = x^2 - 1$, and determine their stability by computing the eigenvalues of the linearized systems.
- (13) Indicate whether each of the following statements is true or false.
- The set of solutions $(x, y, z) \in \mathbb{R}^3$ to the equation $x + y + z = 0$ is a vector subspace of \mathbb{R}^3 of dimension 2.
 - The set of solutions $(x, y, z) \in \mathbb{R}^3$ to the equation $x + y = 1$ is a vector subspace of \mathbb{R}^3 of dimension 2.
 - The set of solutions to the differential equation $y'' + xy' + x^2y = 0$ is a vector space of dimension 2.

- (d) The set of solutions $(x, y, z) \in \mathbb{R}^3$ of the system below is a vector subspace of \mathbb{R}^3 of dimension 1.

$$\begin{aligned}x + 2y + 3z &= 0 \\4x + 5y + 6z &= 0 \\7x + 8y + 9z &= 0\end{aligned}$$

- (e) The polynomials $1 + x$, $1 - x$, $1 + x^2$ are a basis for the vector space of polynomials with real coefficients of degree less than or equal to 2.

- (14) Three identical, well-stirred tanks of with 100 liters of water in each tank are connected in series with tank 1 pumping 10 liter/minute into tank 2, tank 2 pumping 10 liter/minute into tank 3, and tank 3 pumping 10 liter/minute into tank 1. If tank 1 initially has 500 grams of salt dissolved in it, and the other two tanks start at time $t = 0$ with no salt, which of the following initial value problems describes the amounts of salt in grams in each tank ($x_1 =$ salt in tank 1, $x_2 =$ salt in tank 2, $x_3 =$ salt in tank 3).

(a) $x'_1 = \frac{1}{10}x_3 - \frac{1}{10}x_1$ $x'_2 = \frac{1}{10}x_1 - \frac{1}{10}x_2$ $x'_3 = \frac{1}{10}x_2 - \frac{1}{10}x_3$
 $x_1(0) = 500, x_2(0) = 0, x_3(0) = 0.$

(b) $x'_1 = \frac{1}{100}x_2 - \frac{1}{10}x_1$ $x'_2 = \frac{1}{100}x_3 - \frac{1}{10}x_2$ $x'_3 = \frac{1}{100}x_1 - \frac{1}{10}x_3$
 $x_1(0) = 500, x_2(0) = 0, x_3(0) = 0.$

(c) $x'_1 = \frac{1}{10}x_3 - \frac{1}{10}x_1$ $x'_2 = \frac{1}{10}x_1 - \frac{1}{10}x_2$ $x'_3 = \frac{1}{10}x_2 - \frac{1}{10}x_3$
 $x_1(0) = 500, x_2(0) = 0, x_3(0) = 0.$

(d) $x'_1 = \frac{1}{10}x_3 + \frac{1}{10}x_1$ $x'_2 = \frac{1}{10}x_1 + \frac{1}{10}x_2$ $x'_3 = \frac{1}{10}x_2 + \frac{1}{10}x_3$
 $x_1(0) = 500, x_2(0) = 0, x_3(0) = 0.$