

Math 3280 Practice Final Solutions

This is longer than the actual exam, which will be 7-8 questions. You are allowed up to two sheets of notes (both sides) and a calculator, although any use of a calculator must be indicated. Please let me know if you find any typos in these solutions.

- (1) Find the general solution to $(1+t)y' + y = \cos t$.

Solution: In standard form $(y' + P(t)y = Q(t))$ we have $y' + \frac{1}{1+t}y = \frac{\cos(t)}{1+t}$. Using the integrating factor method (section 1.5), we have

$$\rho(t) = e^{\int P(t)dt} = e^{\log(1+t)} = 1+t.$$

Then $\int \rho Q dt = \int \cos t dt = \sin t$ and

$$y = \frac{C}{\rho} + \frac{1}{\rho} \int \rho Q dt = \frac{C}{1+t} + \frac{\sin t}{1+t}.$$

- (2) Rewrite the initial value problem $y''' + y'' + y = t$, $y(0) = y'(0) = y''(0) = 0$ as an equivalent first-order system.

Solution: Introduce the variables $v_1 = y'$, $v_2 = v_1' = y''$ and the system becomes:

$$\begin{aligned} y' &= v_1 \\ v_1' &= v_2 \\ v_2' &= t - v_2 - y \\ y(0) &= 0, \quad v_1(0) = 0, \quad v_2(0) = 0 \end{aligned}$$

Note that rewriting the initial conditions is a required part of this answer.

- (3) Find the general solution to the system

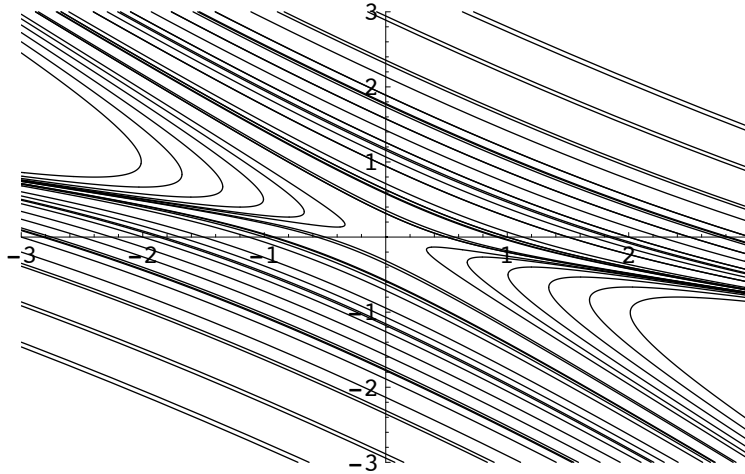
$$\frac{d}{dt} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 2 & 4 \\ -1 & -3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}.$$

Solution: The eigenvalues of the matrix are found from

$$\det \begin{bmatrix} 2-\lambda & 4 \\ -1 & -3-\lambda \end{bmatrix} = \lambda^2 + \lambda - 2 = (\lambda - 1)(\lambda + 2) = 0$$

From row-reducing $A - \lambda I$ for each of these two eigenvalues ($\lambda = 1$ and $\lambda = -2$) we can find that the eigenvectors are $\vec{v}_1 = (-4, 1)$ and $\vec{v}_2 = (-1, 1)$, so the solutions are $x_1 = -4C_1e^t - C_2e^{-2t}$ and $x_2 = C_1e^t + C_2e^{-2t}$.

For large t , $(x_1, x_2) \approx e^t(-4C_1, C_1)$. For large $-t$, $(x_1, x_2) \approx e^{-2t}(-C_2, C_2)$. Some trajectories are shown below.



- (4) Are the vectors $v_1 = (1, 2, 3, 4)$, $v_2 = (2, -2, 4, 2)$, and $v_3 = (0, -3, -1, -3)$ linearly independent? If not, write one of them as a linear combination of the other two.

Solution: The vectors are linearly dependent if there are c_1, c_2, c_3 , not all zero, such that $c_1v_1 + c_2v_2 + c_3v_3 = 0$. This is equivalent to the coefficient

matrix $A = \begin{bmatrix} 1 & 2 & 0 \\ 2 & -2 & -3 \\ 3 & 4 & -1 \\ 4 & 2 & -3 \end{bmatrix}$ having less than 3 pivots after row-reduction. If

we row-reduce A we find

$$\begin{bmatrix} 1 & 2 & 0 \\ 2 & -2 & -3 \\ 3 & 4 & -1 \\ 4 & 2 & -3 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 2 & 0 \\ 0 & -6 & -3 \\ 0 & -2 & -1 \\ 0 & -6 & -3 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 2 & 0 \\ 0 & 1 & \frac{1}{2} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & \frac{1}{2} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

This only has two pivots. The free variable is c_3 , which we can choose to be 2 (to avoid fractions - it would be OK to set it to any nonzero value), which gives $c_2 = -1$ and $c_1 = 2$.

So $2v_1 - v_2 + 2v_3 = 0$; we can write any of the vectors in terms of the other two but the easiest choice here is $v_2 = 2v_1 + 2v_3$.

- (5) Solve the initial value problem $y'' + y = \cos x$, $y'(0) = 0$, $y(0) = -\frac{1}{2}$.

Solution: This could be also done with a Laplace transform. Using undetermined coefficients we find the solution by decomposing it into $y = y_h + y_p$.

The homogeneous solution y_h is found from the characteristic equation $r^2 + 1 = (r - i)(r + i) = 0$ to be $y_h = C_1 \cos(x) + C_2 \sin(x)$.

Since the right-hand side $\cos(x)$ is contained in the solution space of the homogeneous equation, we are forced to consider particular solutions of the form $y_p = Ax \cos(x) + Bx \sin(x)$. Then $y_p'' = -Ax \cos(x) - 2A \sin(x) + 2B \cos(x) - Bx \sin(x)$. Substituting these forms into our ODE yields $-2A \sin(x) + 2B \cos(x) = \cos(x)$, so $A = 0$ and $B = 1/2$.

So now we know that $y = C_1 \cos(x) + C_2 \sin(x) + x \sin(x)/2$. The initial conditions become $C_2 = 0$ and $C_1 = -\frac{1}{2}$, so $y = \frac{-\cos(x) + x \sin(x)}{2}$.

- (6) Use Euler's, the Improved Euler's, or the Runge-Kutta method to numerically approximate $y(2)$ to two digits of accuracy if $y' = t + \sqrt{y}$ and $y(0) = 1$.

Solution: It takes 76 steps to get the desired accuracy with Euler's Method (so this is somewhat harder than anything I would require on the actual final exam). For the improved Euler's method, 5 steps are needed. Fourth-order Runge-Kutta works in 1 step (stepsize 2), giving $y(2) \approx 6.37$ which agrees with $y(2) = 6.411474127809772838513 \dots$ in the first two digits after rounding:

$$f(x, y) = x + \sqrt{y}, \quad h = 2, \quad x_0 = 0, \quad y_0 = 1$$

$$k_1 = f(x_0, y_0) = f(0, 1) = 1$$

$$k_2 = f(x_0 + h/2, y_0 + hk_1/2) = f(1, 2) = 1 + \sqrt{2} \approx 2.41421356$$

$$k_3 = f(x_0 + h/2, y_0 + hk_2/2) = f(1, \sqrt{2} + 2) = \sqrt{\sqrt{2} + 2} + 1 \approx 2.84775907$$

$$k_4 = f(x_0 + h, y_0 + hk_3) = f(2, 2\sqrt{\sqrt{2} + 2} + 3) = \sqrt{2\sqrt{\sqrt{2} + 2} + 3} + 2 \approx 4.58756993$$

$$y(2) \approx y_1 = y_0 + \frac{h}{6}(k_1 + 2k_2 + 2k_3 + k_4) = \sqrt{\sqrt{2} + 2} + \frac{1}{3} \sqrt{2\sqrt{\sqrt{2} + 2} + 3} + \frac{2}{3} \sqrt{2} + \frac{10}{3} \approx 6.37050506$$

It is not necessary to keep the intermediate calculations in exact form, as done above, but you do need to be careful to include enough digits to avoid rounding error.

- (7) Find the general solution to the system

$$\frac{d}{dt} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 & -5 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}.$$

Solution: The characteristic equation is $\det(A - \lambda I) = \lambda^2 - 4\lambda + 8$ with roots (eigenvalues) $\lambda = 2 \pm 2i$. We need to find one eigenvector, let's find it for

$\lambda = 2 + 2i$. We now row reduce

$$A - (2 + 2i)I = \begin{bmatrix} -1 - 2i & -5 \\ 1 & 1 - 2i \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 1 - 2i \\ 0 & 0 \end{bmatrix}$$

So the eigenvector can be chosen to be $v = (-1 + 2i, 1)$. Then the solution to the system is

$$\begin{aligned} x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} &= C_1 \operatorname{Re}[ve^{2t}(\cos(2t) + i \sin(2t))] + C_2 \operatorname{Im}[ve^{2t}(\cos(2t) + i \sin(2t))] \\ &= C_1 \begin{bmatrix} -e^{2t}(\cos(2t) + 2 \sin(2t)) \\ e^{2t} \cos(2t) \end{bmatrix} + C_2 \begin{bmatrix} e^{2t}(-\sin(2t) + 2 \cos(2t)) \\ e^{2t} \sin(2t) \end{bmatrix} \end{aligned}$$

- (8) Find the Laplace transform $X(s) = \mathcal{L}(x(t))$ if $x'' + 8x' + 15x = 0$ and $x(0) = 0$, $x'(0) = 1$. Then find the solution $x(t)$.

Solution: Taking the Laplace transform of the ODE gives

$$\begin{aligned} s^2 X(s) + 8sX(s) + 15X(s) - 8x(0) - sx(0) - x'(0) \\ = s^2 X(s) + 8sX(s) + 15X(s) - 1 = 0. \end{aligned}$$

Solving for $X(s)$ and performing a partial fraction decomposition, we get

$$X(s) = \frac{1}{s^2 + 8s + 15} = \frac{1/2}{s + 3} - \frac{1/2}{s + 5}$$

Since $\mathcal{L}^{-1}\left(\frac{1}{s-a}\right) = e^{at}$, we can invert $X(s)$ to get $x(t) = \frac{e^{-3t}}{2} - \frac{e^{-5t}}{2}$.

The use of the Laplace transform will be optional on the final, but for problems with initial conditions such as $x(0) = 0$, $x'(0) = 0$ it can be the easiest method.

- (9) What is the **form** of the general solution to the ODE $y''' - 4y'' + 14y' - 20y = te^t \cos(3t) + t^2$. Hint: one of the roots of the characteristic polynomial of the left-hand side is 2.

Solution:

First we find the homogeneous solution. The characteristic equation can be factored using the hint to get

$$r^3 - 4r^2 + 14r - 20 = (r - 2)(r^2 - 2r + 10)$$

and then we can use the quadratic equation to get $r = 2, 1 \pm 3i$. So the homogeneous solutions is $y_h = C_1 e^t \sin(3t) + C_2 e^t \cos(3t) + C_3 e^{2t}$.

If there were no overlap with the homogeneous solution we would use the form

$$Ate^t \cos(3t) + Bte^t \sin(3t) + Ce^t \cos(3t) + De^t \sin(3t) + Et^2 + Ft + G$$

for the particular solution, but the terms with C and D are contained in the homogeneous solution so we multiply everything involving this root (the A, B, C , and D terms) by t to get the form of the particular solution:

$$y_p = At^2e^t \cos(3t) + Bt^2e^t \sin(3t) + Cte^t \cos(3t) + Dte^t \sin(3t) + Et^2 + Ft + G.$$

The form of the general solution is the sum of these, $y = y_h + y_p$.

- (10) Consider a mass-spring system with two masses of mass m_1 and m_2 . Mass 1 is connected to a wall with a spring of stiffness k_1 and to mass 2 with a spring of stiffness k_2 . Mass 2 is connected to a second wall with a spring of stiffness k_3 , as shown below. Their displacements from the equilibrium are x_1 and x_2 , which we will combine into a vector $x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$. Then if $x'' = Ax$, show that the eigenvalues of A must be negative if the masses and spring constants are positive.

Solution: As discussed in chapter 7.4, the matrix A has the form:

$$\begin{pmatrix} -(k_1 + k_2)/m_1 & k_2/m_1 \\ k_2/m_2 & -(k_2 + k_3)/m_2 \end{pmatrix}$$

One way to see that the eigenvalues are negative if the k_i and m_i are positive is to use the fact that if λ_1 and λ_2 are the eigenvalues of A , then $\text{tr}(A) = \lambda_1 + \lambda_2$ and $\det(A) = \lambda_1\lambda_2$. The determinant can be simplified to

$$\det(A) = \frac{k_1k_2 + k_1k_3 + k_2k_3}{m_1m_2}$$

which is clearly positive. This means the eigenvalues must have the same sign, but since their trace is negative the sign of each must be negative.

- (11) Use either the Laplace transform method or the eigenvalue/eigenvector method to find the steady state solution to the initial value problem $x' = -x - z$, $y' = -x - y$, $z' = 2x + z$, $x(0) = 0$, $y(0) = 0$, $z(0) = 2$.

Solution: Using the eigenvalue/eigenvector method we first compute the eigenvalues of the coefficient matrix $A = \begin{pmatrix} -1 & 0 & -1 \\ -1 & -1 & 0 \\ 2 & 0 & 1 \end{pmatrix}$ from the characteristic equation $\det(A - \lambda I) = 0$. This factors as $(\lambda + 1)(\lambda^2 + 1) = 0$, so the eigenvalues are $\pm i$ and -1 .

Now we find the eigenvectors. For $\lambda = -1$ we row reduce

$$A + I = \begin{pmatrix} 0 & 0 & -1 \\ -1 & 0 & 0 \\ 2 & 0 & 2 \end{pmatrix}$$

to get $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$. The kernel of this matrix consists of vectors of the form $\begin{pmatrix} 0 \\ a \\ 0 \end{pmatrix}$ for any a . We can choose $a = 1$.

For the complex conjugate pair we can use either eigenvalue. If we choose to use i , then we row reduce $A - iI$ to get $\begin{pmatrix} 1 & 0 & 1/2 - i/2 \\ 0 & 1 & i/2 \\ 0 & 0 & 0 \end{pmatrix}$. If we choose the last entry of the eigenvalue to be 2 , the eigenvector is $\begin{pmatrix} -1 + i \\ -i \\ 2 \end{pmatrix}$.

The general solution is

$$\begin{aligned} \begin{pmatrix} x \\ y \\ z \end{pmatrix} &= C_1 \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} e^{-t} + C_2 \operatorname{Re} \left(\begin{pmatrix} -1 + i \\ -i \\ 2 \end{pmatrix} (\cos(t) + i \sin(t)) \right) + C_3 \operatorname{Im} \left(\begin{pmatrix} -1 + i \\ -i \\ 2 \end{pmatrix} (\cos(t) + i \sin(t)) \right) \\ &= C_1 \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} e^{-t} + C_2 \begin{pmatrix} -\cos(t) - \sin(t) \\ \sin(t) \\ 2\cos(t) \end{pmatrix} + C_3 \begin{pmatrix} \cos(t) - \sin(t) \\ -\cos(t) \\ 2\sin(t) \end{pmatrix}. \end{aligned}$$

Now we can use the initial conditions; evaluating at $t = 0$ gives $-C_2 + C_3 = 0$, $C_1 - C_3 = 0$, and $2C_2 = 2$. So $C_3 = 1$ and $C_1 = 1$. For the steady state solution we drop the first term since e^{-t} will decay to 0. So the steady state solution is:

$$\begin{pmatrix} -2\sin(t) \\ -\cos(t) + \sin(t) \\ 2\cos(t) + 2\sin(t) \end{pmatrix}$$

- (12) Find the equilibria of the system $x' = 2y^3 - 2x$, $y' = x^2 - 1$, and determine their stability by computing the eigenvalues of the linearized systems.

Solution: To find the equilibria we solve the pair of equations $2y^3 - 2x = 0$, $x^2 - 1 = 0$. The second equation is simpler, since it only involves x - any equilibria must have $x = \pm 1$. Substituting these values into the first equation gives $y^3 = \pm 1$, so $y = \pm 1$ and y is the same sign as x . I.e. the two equilibria are $(1, 1)$ and $(-1, -1)$.

The Jacobian matrix of the functions $f_1(x, y) = 2y^3 - 2x$ and $f_2 = x^2 - 1$ is

$$\begin{pmatrix} \frac{\partial f_1}{\partial x} & \frac{\partial f_1}{\partial y} \\ \frac{\partial f_2}{\partial x} & \frac{\partial f_2}{\partial y} \end{pmatrix} = \begin{pmatrix} \frac{\partial(2y^3-2x)}{\partial x} & \frac{\partial(2y^3-2x)}{\partial y} \\ \frac{\partial(x^2-1)}{\partial x} & \frac{\partial(x^2-1)}{\partial y} \end{pmatrix} = \begin{pmatrix} -2 & 6y^2 \\ 2x & 0 \end{pmatrix}$$

At the equilibrium $(1, 1)$ this becomes $\begin{pmatrix} -2 & 6 \\ 2 & 0 \end{pmatrix}$. The eigenvalues are solutions of $\lambda^2 + 2\lambda - 12 = 0$, which are $-1 \pm \sqrt{13}$. Since $-1 + \sqrt{13} > 0$, there is a positive eigenvalue and the equilibrium is unstable.

At the equilibrium $(-1, -1)$ the Jacobian becomes $\begin{pmatrix} -2 & 6 \\ -2 & 0 \end{pmatrix}$. The eigenvalues are solutions of $\lambda^2 + 2\lambda + 12 = 0$, which are $-1 \pm \sqrt{13}i$. Since the real parts of these are negative, the equilibrium is stable (nearby solutions would spiral inwards).

(13) Indicate whether each of the following statements is true or false.

- (a) The set of solutions $(x, y, z) \in \mathbb{R}^3$ to the equation $x + y + z = 0$ is a vector subspace of \mathbb{R}^3 of dimension 2.

Solution: True. A single linear homogeneous constraint will have a solution set that is one dimension less than the ambient vector space. Alternatively we can compute this by row-reducing the coefficient matrix of the system, which in this case is the matrix $[1, 1, 1]$. This is already in row-reduced echelon form, with one pivot and two free variables (y and z). The number of free variables is the dimension of the solution set.

- (b) The set of solutions $(x, y, z) \in \mathbb{R}^3$ to the equation $x + y = 1$ is a vector subspace of \mathbb{R}^3 of dimension 2.

Solution: False. This is a nonhomogeneous system, so the solutions do not form a vector subspace.

- (c) The set of solutions to the differential equation $y'' + xy' + x^2y = 0$ is a vector space of dimension 2.

Solution: True. See Theorem 4 of section 5.2.

- (d) The set of solutions $(x, y, z) \in \mathbb{R}^3$ of the system below is a vector subspace of \mathbb{R}^3 of dimension 1.

$$\begin{aligned} x + 2y + 3z &= 0 \\ 4x + 5y + 6z &= 0 \\ 7x + 8y + 9z &= 0 \end{aligned}$$

Solution: True. The coefficient matrix has a row-reduced form with two pivots and one free variable.

- (e) The polynomials $1 + x$, $1 - x$, $1 + x^2$ are a basis for the vector space of polynomials with real coefficients of degree less than or equal to 2.

Solution: True. A more obvious basis would be $1, x, x^2$, which can be obtained from these polynomials as linear combinations: $1 = (1 + x)/2 + (1 - x)/2$, $x = (1 + x)/2 - (1 - x)/2$, and $x^2 = -(1 + x)/2 - (1 - x)/2 + (1 + x^2)$.

- (14) Three identical, well-stirred tanks of with 100 liters of water in each tank are connected in series with tank 1 pumping 10 liter/minute into tank 2, tank 2 pumping 10 liter/minute into tank 3, and tank 3 pumping 10 liter/minute into tank 1. If tank 1 initially has 500 grams of salt dissolved in it, and the other two tanks start at time $t = 0$ with no salt, which of the following initial value problems describes the amounts of salt in grams in each tank ($x_1 =$ salt in tank 1, $x_2 =$ salt in tank 2, $x_3 =$ salt in tank 3).

Solution: Answer (c) is correct

$$x_1' = \frac{1}{10}x_3 - \frac{1}{10}x_1 \quad x_2' = \frac{1}{10}x_1 - \frac{1}{10}x_2 \quad x_3' = \frac{1}{10}x_2 - \frac{1}{10}x_3$$