Math 3298 Practice Final Solutions

Please let me know if you find any mistakes or typos.

(1) Find the integral of the function $f(x,y) = 2x\sqrt{y^2-x^2}$ over the triangle $T = \{(x,y) \mid 0 \le y \le 2, 0 \le x \le y\}$

Solution: This can be done in either order but its easier to do the x-integral first:

$$\int_0^2 \int_0^y 2x\sqrt{y^2-x^2}dxdy = -\frac{2}{3}\int_0^2 (y^2-x^2)^{3/2}|_0^y dy$$

$$= \frac{2}{3}\int_0^2 y^3 dy = \frac{y^4}{6}|_0^2 = 8/3$$

The first integral is done with a substitution $u = y^2 - x^2$.

(2) Find the volume of the solid inside the sphere $x^2+y^2+z^2 = 9$ and outside the cylinder $x^2+y^2 = 1$.

Solution: This is probably easiest in cylindrical coordinates. Solving the sphere boundary equation for z we find $z = \pm\sqrt{9-x^2-y^2} = \pm\sqrt{9-r^2}$. So the volume is

$$\int_0^{2\pi}\int_1^3\int_{-\sqrt{9-r^2}}^{\sqrt{9-r^2}} rdzdrd\theta = \int_0^{2\pi}\int_1^3 2r\sqrt{9-r^2}drd\theta$$

$$= \int_0^{2\pi} -\frac{2}{3}(9-r^2)^{3/2}|_1^3 d\theta = \frac{4\pi\ 8^{3/2}}{3}$$

(3) Compute the integral $\int\int\int_R \sqrt{x^2+y^2}\ dV$ where R is the region inside the cylinder $x^2+y^2 = 25$ and between $z=-1$ and $z=4$.

Solution: Again, cylindrical coordinates are the best choice for this problem.

$$\int\int\int_R \sqrt{x^2+y^2}\ dV = \int_0^5\int_0^{2\pi}\int_{-1}^4 r^2\ dz\ d\theta\ dr = \int_0^5\int_0^{2\pi} 5r^2\ d\theta\ dr$$

$$= \int_0^5 10\pi r^2\ dr = \frac{10\pi r^3}{3}|_0^5 = \frac{1250\pi}{3}$$

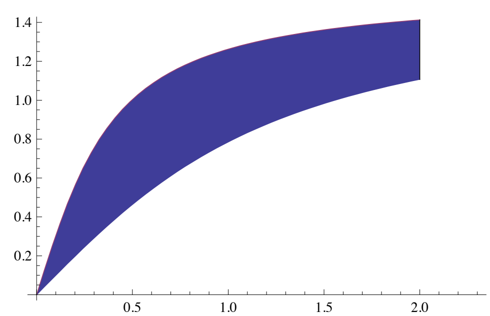
(4) Find the volume of the solid bounded by the planes $z = x$, $y = x$, $x + y = 2$, and $z = 0$.

Solution: This is a tetrahedron. By considering any three of the four boundary equations, we can find that the vertices are $(0, 0, 0)$, $(1, 1, 0)$, $(1, 1, 1)$, and $(0, 2, 0)$. This helps to sketch the figure and determine the bounds for the integral:

$$V = \int_0^1 \int_x^{2-x} \int_0^x dz\ dy\ dx = \int_0^1 \int_x^{2-x} x\ dy\ dx = \int_0^1 xy|_x^{2-x}\ dx$$
$$= \int_0^1 (2x - 2x^2)\ dx = 1/3$$

(5) Change the order of integration of $\int_0^2 \int_{Arctan(x)}^{Arctan(\pi x)} dydx$ and evaluate the integral.

Solution: This question is definitely a bit harder than one I would put on an exam. The integration region is shown below.



To do the x-integral first we need to split up the region into two pieces because of the corner at $(2, \arctan(2))$. Then we have

$$\int_0^{\arctan(2)} \int_{\tan(y)/\pi}^{tan(y)} dx\ dy + \int_{\arctan(2)}^{\arctan(2\pi)} \int_{\tan(y)/\pi}^{2} dx\ dy =$$

$$\int_0^{\arctan(2)} (\tan(y) - \tan(y)/\pi)dy + \int_{\arctan(2)}^{\arctan(2\pi)} (2 - \tan(y)/\pi)dy =$$

$$-2\arctan(2) + 2\arctan(2\pi) + \frac{(-1+\pi)\log(5)}{2\pi} + \frac{\log\left(\frac{5}{1+4\pi^2}\right)}{2\pi} \approx .827\ldots$$

The final answer has been simplified using several properties of the logarithm: $\log(a) - \log(b) = \log(a/b)$, $\log(1/b) = -\log(b)$, and $\log(a^b) = b\log(a)$.

(6) Compute the integral $\int_{-1}^{1}\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}}\int_{x^2+y^2}^{2-x^2-y^2}(x^2+y^2)^{3/2}\,dz\,dy\,dx$ by changing to cylindrical coordinates.

Solution: The projection of the region onto the x-y plane is the disk of radius 1. So the integral can be rewritten in cylindrical coordinates as:

$$\int_0^{2\pi}\int_0^1\int_{r^2}^{2-r^2} r^4\,dz\,dr\,d\theta$$

which evaluates to

$$= \frac{1}{5}\int_0^{2\pi}\int_0^1 (2-2r^2)r^4\,dr\,d\theta = 8\pi/35$$

(7) This difficult a problem would be extra credit: Assuming that $\beta \in (0, \pi/2)$ and $a > 0$, compute the following integral

$$\int_0^{a\sin\beta}\int_{y\cot\beta}^{\sqrt{a^2-y^2}}\int_0^1 \ln(x^2+y^2)\,dz\,dx\,dy$$

Solution: The z-integral is easy and we get

$$\int_0^{a\sin\beta}\int_{y\cot\beta}^{\sqrt{a^2-y^2}} \ln(x^2+y^2)\,dx\,dy$$

The first thing to do is understand the region of integration. The upper x-boundary $x = \sqrt{a^2-y^2}$ is the right-hand semicircle of radius a centered at $(0,0)$. The lower x-boundary is the line $x = \cot(\beta)y$ or $y = \tan(\beta)x$, a line through $(0,0)$ with angle β. The y boundaries are the x-axis and the height where the line intersects the circle. So our region of integration is simply a circular wedge of radius a and angle β from the x-axis. Then our integral is much easier in polar coordinates:

$$\int_0^a\int_0^\beta \ln(r^2)r\,d\theta\,dr = \beta\int_0^a \ln(r^2)r\,dr$$

With a substitution $u = r^2$, this integral can be done with integration by parts, or looked up in a table, with the final answer being $\beta a^2(\ln(a) - \frac{1}{2})$.

(8) Reverse the order of integration for the integral $\int_0^1\int_x^1\int_0^{y^2} f(x,y,z)dzdydx$.

Solution: The answer is:

$$\int_0^1 \int_{\sqrt{z}}^1 \int_0^y f(x, y, z)\ dx\ dy\ dz.$$

The projection onto the xy plane is the triangle with vertices $(0, 0)$, $(1, 1)$ and $(0, 1)$. The surface slopes up in the z-direction parabolically to the line $y = 1$.

(9) Compute the vector line integral $\int_C \vec{F} \cdot d\vec{r}$ where C is the path $(4-3^t, -2+2t, \pi t)$, $t \in [0, 1]$, and $\vec{F} = (2x \cos z - x^2, z - 2y, y - x^2 \sin z)$.

Solution: The field is conservative since $curl(\vec{F}) = \vec{0}$. A potential function for $\vec{F}$ is $f = x^2 \cos z - x^3/3 + yz - y^2$, so $\int_C \vec{F} \cdot d\vec{r} = f(1, 0, \pi) - f(3, -2, 0) = -4/3 - (9 - 9 - 4) = 8/3$.

(10) Find the linearization of $f(x, y)$ at $(x, y) = (0, 1)$ if $f = h(u(x, y), v(x, y))$ and $\text{grad}(h)|_{(1,1)} = (\frac{\partial h}{\partial u}, \frac{\partial h}{\partial v})|_{(1,1)} = (2, 3)$, $u(x, y) = x + y$, and $v(x, y) = y^2$.

Solution: The linearization is $L(x, y) = f(0, 1) + \frac{\partial f}{\partial x}|_{(0,1)}(x - 0) + \frac{\partial f}{\partial y}|_{(0,1)}(y - 1)$. To find the partial derivatives we use the chain rule and the derivatives $\frac{\partial u}{\partial x}| = 1$, $\frac{\partial u}{\partial y} = 1$, $\frac{\partial v}{\partial x}| = 0$, $\frac{\partial v}{\partial y}|_{0,1} = 2$:

$$\frac{\partial f}{\partial x}|_{(0,1)} = \frac{\partial h}{\partial u}|_{(1,1)} \frac{\partial u}{\partial x}|_{(0,1)} + \frac{\partial h}{\partial v}|_{(1,1)} \frac{\partial v}{\partial x}|_{(0,1)} = 2 \cdot 1 + 3 \cdot 0 = 2$$

$$\frac{\partial f}{\partial y}|_{(0,1)} = \frac{\partial h}{\partial u}|_{(1,1)} \frac{\partial u}{\partial y}|_{(0,1)} + \frac{\partial h}{\partial v}|_{(1,1)} \frac{\partial v}{\partial y}|_{(0,1)} = 2 \cdot 1 + 3 \cdot 2 = 8$$

So

$$L(x, y) = f(0, 1) + 2(x - 0) + 8(y - 1) = f(0, 1) + 2x + 8y - 8.$$

Since neither f nor h is given explicitly this is all that is possible to determine, besides the fact that $f(0, 1) = h(1, 1)$.

(11) Find the surface area of the torus parameterized by $x = (2 + \cos(v)) \cos(u)$, $y = (2 + \cos(v)) \sin(u)$, $z = \sin(v)$, with $u \in [0, 2\pi]$ and $v \in [0, 2\pi]$.

Solution: The surface area element is computed from the length of the cross product of the partials $\frac{\partial \vec{r}}{\partial u}$ and $\frac{\partial \vec{r}}{\partial v}$, where $\vec{r} = (x, y, z)$. After many uses of the identity $\sin^2(t) + \cos^2(t) = 1$, this simplifies to

$$dS = |\frac{\partial \vec{r}}{\partial u} \times \frac{\partial \vec{r}}{\partial v}|dudv = |2 + \cos(v)|dudv.$$

The absolute value function can be dropped because $2 + \cos(v) > 0$. Now the surface area can be computed:

$$S.A. = \int_0^{2\pi} \int_0^{2\pi} (2 + \cos(v))dudv = 2\pi \int_0^{2\pi} (2 + \cos(v))dv = 8\pi^2.$$

(12) Find the maxima and minima of $f(x, y) = \frac{1}{x} + \frac{2}{y}$ on the set $\frac{1}{x^2} + \frac{1}{y^2} = 1$.

Solution: I will use the Lagrange multiplier method. Let $g = \frac{1}{x^2} + \frac{1}{y^2} - 1$ (the constraint), and then require that $\bigtriangledown(f) = \lambda \bigtriangledown(g)$, i.e.

$$(-\frac{1}{x^2}, -\frac{2}{y^2}) = (-\frac{2\lambda}{x^3}, -\frac{2\lambda}{y^3}).$$

After clearing denominators we find that $\lambda = x/2 = y$. Using that relation between x and y, the constraint equations becomes $g(x, x/2) = \frac{1}{x^2} + \frac{4}{x^2} - 1 = 0$ or $x^2 = 5$. So there are two critical points on the constraint curve, $\pm(\sqrt{5}, \sqrt{5}/2)$. Comparison with other values of f on the constraint curve shows that $f(\sqrt{5}, \sqrt{5}/2) = \sqrt{5}$ is a maximum and $f(-\sqrt{5}, -\sqrt{5}/2) = -\sqrt{5}$ is a minimum.

(13) Find the volume of the solid wedge bounded by the planes $z = 0$ and $z = -2y$ and the cylinder $x^2 + y^2 = 4$ (with $y \geq 0$).

Solution: In cylindrical coordinates

$$V = \int_0^{\pi} \int_0^{2} \int_{-2r\sin\theta}^{0} dz\ r\ dr\ d\theta = \int_0^{\pi} \int_0^{2} 2r^2 \sin\theta\ dr\ d\theta$$

$$= \int_0^{\pi} 16\sin\theta/3\ d\theta = 32/3.$$

(14) Use Green's Theorem to find the smooth, simple, closed and positively oriented curve in the plane for which the line integral $\oint(\frac{x^2y}{4} + \frac{y^3}{3})dx + xdy$ has the largest possible value.

Solution: The corresponding double integral from Green's theorem is

$$\int\int_R (1 - x^2/4 - y^2)\ dA.$$

The integrand is positive in the interior of the ellipse $x^2/4+y^2 = 1$, so we choose this ellipse as the desired curve ($x = 2\cos(t)$, $y = \sin t$).

(15) Compute the value of $\int\int_S (\nabla \times \vec{F}) \cdot \vec{n}\ dS$ where S is the upper half of the ellipsoid $4x^2 + 9y^2 + 36z^2 = 36$, $z \geq 0$, with upward pointing normal, and $\vec{F} = (y, x^2, (x^2 + y^2)^{3/2} e^{xyz})$.

Solution: The presence of nasty stuff like e^{xyz} inspires us to reformulate the computation using Stokes' theorem. The ellipse boundary can be parameterized as $x = 3\cos t$, $y = 2\sin t$, $z = 0$. The corresponding line integral is then

$$\oint_C \vec{F} \cdot d\vec{r} = \int_0^{2\pi} (2\sin t, 9\cos^2 t, (9\cos^2(t) + 4\sin^2(t))^{3/2}) \cdot (-3\sin t, 2\cos t, 0) dt =$$

$$\int_0^{2\pi} -6\sin^2(t) + 18\cos^3(t)\ dt = -6\pi.$$

(16) Let $\vec{r}(t)$ be a curve in space with unit tangent, normal, and binormal vectors $\vec{T}$, $\vec{N}$, and $\vec{B}$. Show that $\frac{d\vec{B}}{dt}$ is perpendicular to $\vec{T}$.

Solution: Since $\vec{B} = \vec{T} \times \vec{N}$, $\frac{d\vec{B}}{dt} = \frac{d\vec{T}}{dt} \times \vec{N} + \vec{T} \times \frac{d\vec{N}}{dt}$. However $\vec{N}$ is parallel to $\frac{d\vec{T}}{dt}$, so in fact $\frac{d\vec{B}}{dt} = \vec{T} \times \frac{d\vec{N}}{dt}$. Finally, note that $\vec{a}$ is always perpendicular to $\vec{a} \times \vec{b}$ for any vectors $\vec{a}$, $\vec{b}$.

(17) Compute the flux integral $\int\int_S \vec{F} \cdot \vec{n}\ dS$ where S is the graph of $z = 1 - x^2 - y^2$, with upward normal, for $z \geq 0$, and with $\vec{F} = (xz, yz, 2z^2)$.

Solution: The given flux integral can be computed directly or by using the divergence theorem. I will do it both ways for comparison (on a test, one method would be sufficient). Directly: since the surface is given as a graph $z = f(x, y)$, $\vec{n}\ dS = d\vec{S} = (-f_x, -f_y, 1) = (2x, 2y, 1)$. Then

$$\int\int_S \vec{F} \cdot \vec{n}\ dS = \int\int_S (xz, yz, 2z^2)|_S \cdot (2x, 2y, 1)\ dxdy$$

$$= \int\int_S (1-x^2-y^2)(2x^2+2y^2)+2(1-x^2-y^2)^2\ dxdy = 2\int_0^{2\pi}\int_0^1 (1-r^2)(2r^2+2(1-r^2))r\ drd\theta$$
$$= 4\int_0^{2\pi}\int_0^1 (r-r^3)\ drd\theta = \pi.$$

We cannot immediately apply the divergence theorem because this surface is not closed. However, we could consider the closed surface $S_2 = S \cup S_1$, where S_1 is the unit disk $z = 0$, $x^2 + y^2 \le 1$ and with normal $(0, 0, -1)$. Since $\vec{F} = 0$ on S_1, this addition doesn't actually affect the flux integral, i.e. $\int\int_{S_2} \vec{F}\cdot\vec{n}\ dS_2 = \int\int_S \vec{F}\cdot\vec{n}\ dS$ but $\int\int_{S_2} \vec{F}\cdot\vec{n}\ dS_2 = \int\int\int div\vec{F} dV$. Since $div\vec{F} = 6z$, the value we are after can be computed in cylindrical coordinates as:

$$\int_0^{2\pi}\int_0^1\int_0^{1-r^2} 6zr\ dzdrd\theta = 3\int_0^{2\pi}\int_0^1\int_0^{1-r^2} (1-r^2)^2 r\ drd\theta = \pi.$$

(18) Use the divergence theorem to compute the flux of $\vec{F} = (z^5 + x, \cos(xz), z^2)$ through the surface bounded by $z = 0$ and $z = 1 - x^2 - y^2$.

Solution: The divergence of $\vec{F}$ is $1 + 2z$, so

$$\int\int_S \nabla \times \vec{F}\cdot d\vec{S} = \int\int\int_R (1+2z)dV$$

where R is the interior of S. To evaluate this it is easiest to use cylindrical coordinates:

$$\int_0^1\int_0^{2\pi}\int_0^{1-r^2} (1+2z)r\ dz\ d\theta\ dr = \int_0^1\int_0^{2\pi} (2r-3r^3+r^5)\ d\theta\ dr = 5/12\int_0^{2\pi}\ d\theta = 5\pi/6$$