(1) Find the integral of the function \( f(x, y) = 2x\sqrt{y^2 - x^2} \) over the triangle \( T = \{(x, y) \mid 0 \leq y \leq 2, 0 \leq x \leq y\} \).

Solution: This can be done in either order but its easier to do the \( x \)-integral first:

\[
\int_0^2 \int_0^y 2x\sqrt{y^2 - x^2} \, dx \, dy = -\frac{2}{3} \int_0^2 (y^2 - x^2)^{3/2} \, dy
\]

\[
= \frac{2}{3} \int_0^2 y^3 \, dy = \frac{y^4}{6} \bigg|_0^2 = \frac{8}{3}
\]

The first integral is done with a substitution \( u = y^2 - x^2 \).

(2) Find the volume of the solid inside the sphere \( x^2 + y^2 + z^2 = 9 \) and outside the cylinder \( x^2 + y^2 = 1 \).

Solution: This is probably easiest in cylindrical coordinates. Solving the sphere boundary equation for \( z \) we find \( z = \pm \sqrt{9 - x^2 - y^2} = \pm \sqrt{9 - r^2} \). So the volume is

\[
\int_0^{2\pi} \int_0^3 \int_{\sqrt{9-r^2}}^{\sqrt{9-r^2}} r \, dz \, dr \, d\theta = \int_0^{2\pi} \int_1^3 2r \sqrt{9-r^2} \, dr \, d\theta
\]

\[
= \int_0^{2\pi} -\frac{2}{3}(9-r^2)^{3/2} \bigg|_1^3 \, d\theta = \frac{4\pi}{3} \frac{8^{3/2}}{3}
\]

(3) Compute the integral \( \int \int \int_R \sqrt{x^2 + y^2} \, dV \) where \( R \) is the region inside the cylinder \( x^2 + y^2 = 25 \) and between \( z = -1 \) and \( z = 4 \).

Solution: Again, cylindrical coordinates are the best choice for this problem.

\[
\int \int \int_R \sqrt{x^2 + y^2} \, dV = \int_0^5 \int_0^{2\pi} \int_{-1}^4 r^2 \, dz \, d\theta \, dr = \int_0^5 \int_0^{2\pi} 5r^2 \, d\theta \, dr
\]

\[
= \int_0^5 10\pi r^3 \, dr = \frac{10\pi r^4}{3} \bigg|_0^5 = \frac{1250\pi}{3}
\]
(4) Find the volume of the solid bounded by the planes \( z = x, \ y = x, \ x + y = 2, \) and \( z = 0. \)

Solution: This is a tetrahedron. By considering any three of the four boundary equations, we can find that the vertices are \((0, 0, 0), (1, 1, 0), (1, 1, 1),\) and \((0, 2, 0).\) This helps to sketch the figure and determine the bounds for the integral:

\[
V = \int_0^1 \int_0^{2-x} \int_0^x dz \ dy \ dx = \int_0^1 \int_0^{2-x} x \ dy \ dx = \int_0^1 xy|_x^{2-x} \ dx = \int_0^1 (2x - 2x^2) \ dx = 1/3
\]

(5) Change the order of integration of \( \int_0^2 \int_{\arctan(\pi x)}^0 dy \ dx \) and evaluate the integral.

Solution: This question is definitely a bit harder than one I would put on an exam. The integration region is shown below.

To do the \( x\)-integral first we need to split up the region into two pieces because of the corner at \((2, \arctan(2)).\) Then we have

\[
\int_0^{\arctan(2)} \int_0^{\tan(y)} \ dx \ dy + \int_0^{\arctan(2\pi)} \int_{\arctan(y)/\pi}^{2} \ dx \ dy = \\
\int_0^{\arctan(2)} \int_0^{\tan(y) - \tan(y)/\pi} \ dx \ dy + \int_0^{\arctan(2\pi)} \int_{\arctan(y)/\pi}^{2 - \tan(y)/\pi} \ dy = \\
-2 \arctan(2) + 2 \arctan(2\pi) + \frac{(1 - \pi) \log(5)}{2\pi} + \frac{\log\left(\frac{5}{1+4\pi^2}\right)}{2\pi} \approx .827 \ldots
\]

The final answer has been simplified using several properties of the logarithm: \( \log(a) - \log(b) = \log(a/b), \log(1/b) = -\log(b), \) and \( \log(a^b) = b \log(a). \)
(6) Compute the integral \[ \int_{-1}^{1} \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} \int_{x^2+y^2}^{2-x^2-y^2} (x^2 + y^2)^{3/2} \, dz \, dy \, dx \] by changing to cylindrical coordinates.

Solution: The projection of the region onto the x-y plane is the disk of radius 1. So the integral can be rewritten in cylindrical coordinates as:

\[ \int_{0}^{2\pi} \int_{0}^{1} \int_{r^2}^{2-r^2} r^4 \, dz \, dr \, d\theta \]

which evaluates to

\[ = \frac{1}{5} \int_{0}^{2\pi} \int_{0}^{1} (2 - 2r^2)r^4 \, dr \, d\theta = \frac{8\pi}{35} \]

(7) This difficult problem would be extra credit: Assuming that \( \beta \in (0, \pi/2) \) and \( a > 0 \), compute the following integral

\[ \int_{0}^{a \sin \beta} \int_{y \cot \beta}^{\sqrt{a^2-y^2}} \int_{0}^{1} \ln(x^2 + y^2) \, dz \, dx \, dy \]

Solution: The z-integral is easy and we get

\[ \int_{0}^{a \sin \beta} \int_{y \cot \beta}^{\sqrt{a^2-y^2}} \ln(x^2 + y^2) \, dx \, dy \]

The first thing to do is understand the region of integration. The upper x-boundary \( x = \sqrt{a^2 - y^2} \) is the right-hand semicircle of radius \( a \) centered at \((0,0)\). The lower x-boundary is the line \( x = \cot(\beta)y \) or \( y = \tan(\beta)x \), a line through \((0,0)\) with angle \( \beta \). The y boundaries are the x-axis and the height where the line intersects the circle. So our region of integration is simply a circular wedge of radius \( a \) and angle \( \beta \) from the x-axis. Then our integral is much easier in polar coordinates:

\[ \int_{0}^{a \sin \beta} \int_{y \cot \beta}^{\sqrt{a^2-y^2}} \ln(x^2 + y^2) \, dx \, dy \]

With a substitution \( u = r^2 \), this integral can be done with integration by parts, or looked up in a table, with the final answer being \( \beta a^2 (\ln(a) - \frac{1}{2}) \).

(8) Reverse the order of integration for the integral \( \int_{0}^{1} \int_{x}^{1} y^2 f(x, y, z) \, dz \, dy \, dx \).
Solution: The answer is:

\[ \int_0^1 \int_{\sqrt{z}}^1 \int_0^y f(x, y, z) \, dx \, dy \, dz. \]

The projection onto the \( xy \) plane is the triangle with vertices \((0, 0), (1, 1)\) and \((0, 1)\). The surface slopes up in the \( z \)-direction parabolically to the line \( y = 1 \).

(9) Compute the vector line integral \( \int_C \vec{F} \cdot d\vec{r} \) where \( C \) is the path \((4 - 3t, -2 + 2t, \pi t)\), \( t \in [0, 1] \), and \( \vec{F} = (2x \cos z - x^2, z - 2y, y - x^2 \sin z) \).

Solution: The field is conservative since \( \text{curl} (\vec{F}) = \vec{0} \). A potential function for \( \vec{F} \) is \( f(x, y, z) = x^2 \cos z - x^3/3 + yz - y^2 \), so \( \int_C \vec{F} \cdot d\vec{r} = f(1, 0, \pi) - f(3, -2, 0) = -4/3 - (9 - 9 - 4) = 8/3 \).

(10) Find the linearization of \( f(x, y) \) at \((x, y) = (0, 1)\) if \( f = h(u(x, y), v(x, y)) \) and \( \text{grad}(h)|_{(1, 1)} = (\frac{\partial h}{\partial u}|_{(1, 1)}, \frac{\partial h}{\partial v}|_{(1, 1)}) = (2, 3) \), \( u(x, y) = x + y \), and \( v(x, y) = y^2 \).

Solution: The linearization is \( L(x, y) = f(0, 1) + \frac{\partial f}{\partial x}|_{(0, 1)}(x - 0) + \frac{\partial f}{\partial y}|_{(0, 1)}(y - 1) \).

To find the partial derivatives we use the chain rule and the derivatives \( \frac{\partial u}{\partial x}|_{0, 1} = 1 \), \( \frac{\partial u}{\partial y}|_{0, 1} = 0 \), \( \frac{\partial v}{\partial x}|_{0, 1} = 2 \):

\[
\frac{\partial f}{\partial x}|_{(0, 1)} = \frac{\partial h}{\partial u}|_{(1, 1)} \frac{\partial u}{\partial x}|_{(0, 1)} + \frac{\partial h}{\partial v}|_{(1, 1)} \frac{\partial v}{\partial x}|_{(0, 1)} = 2 \cdot 1 + 3 \cdot 0 = 2
\]

\[
\frac{\partial f}{\partial y}|_{(0, 1)} = \frac{\partial h}{\partial u}|_{(1, 1)} \frac{\partial u}{\partial y}|_{(0, 1)} + \frac{\partial h}{\partial v}|_{(1, 1)} \frac{\partial v}{\partial y}|_{(0, 1)} = 2 \cdot 1 + 3 \cdot 2 = 8
\]

So

\( L(x, y) = f(0, 1) + 2(x - 0) + 8(y - 1) = f(0, 1) + 2x + 8y - 8. \)

Since neither \( f \) nor \( h \) is given explicitly this is all that is possible to determine, besides the fact that \( f(0, 1) = h(1, 1) \).

(11) Find the surface area of the torus parameterized by \( x = (2 + \cos(v)) \cos(u), y = (2 + \cos(v)) \sin(u), z = \sin(v) \), with \( u \in [0, 2\pi] \) and \( v \in [0, 2\pi] \).

Solution: The surface area element is computed from the length of the cross product of the partials \( \frac{\partial \vec{r}}{\partial u} \) and \( \frac{\partial \vec{r}}{\partial v} \), where \( \vec{r} = (x, y, z) \). After many uses of the
identity $\sin^2(t) + \cos^2(t) = 1$, this simplifies to

\[
dS = \left| \frac{\partial \vec{r}}{\partial u} \times \frac{\partial \vec{r}}{\partial v} \right| dudv = |2 + \cos(v)| dudv.
\]

The absolute value function can be dropped because $2 + \cos(v) > 0$. Now the surface area can be computed:

\[
S.A. = \int_0^{2\pi} \int_0^{2\pi} (2 + \cos(v)) dudv = 2\pi \int_0^{2\pi} (2 + \cos(v)) dv = 8\pi^2.
\]

(12) Find the maxima and minima of $f(x, y) = \frac{1}{x} + \frac{2}{y}$ on the set $\frac{1}{x^2} + \frac{1}{y^2} = 1$.

Solution: I will use the Lagrange multiplier method. Let $g = \frac{1}{x^2} + \frac{1}{y^2} - 1$ (the constraint), and then require that $\nabla(f) = \lambda \nabla(g)$, i.e.

\[
\begin{pmatrix}
-\frac{1}{x^2} \\
-\frac{2}{y^2}
\end{pmatrix} = \begin{pmatrix}
-\frac{2\lambda}{x^3} \\
-\frac{2\lambda}{y^3}
\end{pmatrix}.
\]

After clearing denominators we find that $\lambda = x/2 = y$. Using that relation between $x$ and $y$, the constraint equations becomes $g(x, x/2) = \frac{1}{x^2} + \frac{4}{x^2} - 1 = 0$ or $x^2 = 5$. So there are two critical points on the constraint curve, $x = \pm(\sqrt{5}, \sqrt{5}/2)$. Comparison with other values of $f$ on the constraint curve shows that $f(\sqrt{5}, \sqrt{5}/2) = \sqrt{5}$ is a maximum and $f(-\sqrt{5}, -\sqrt{5}/2) = -\sqrt{5}$ is a minimum.

(13) Find the volume of the solid wedge bounded by the planes $z = 0$ and $z = -2y$ and the cylinder $x^2 + y^2 = 4$ (with $y \geq 0$).

Solution: In cylindrical coordinates

\[
V = \int_0^{\pi} \int_0^2 \int_{-2r\sin\theta}^0 dz \ r \ dr \ d\theta = \int_0^{\pi} \int_0^2 2r^2 \sin\theta \ dr \ d\theta
\]

\[
= \int_0^{\pi} 16 \sin\theta / 3 \ d\theta = 32/3.
\]

(14) Use Green’s Theorem to find the smooth, simple, closed and positively oriented curve in the plane for which the line integral $\int (\frac{x^2}{4} + \frac{y^4}{8})dx + xdy$ has the largest possible value.
Solution: The corresponding double integral from Green’s theorem is

\[ \int \int_R \left(1 - \frac{x^2}{4} - \frac{y^2}{4} \right) \, dA. \]

The integrand is positive in the interior of the ellipse \( \frac{x^2}{4} + \frac{y^2}{4} = 1 \), so we choose this ellipse as the desired curve \((x = 2 \cos(t), y = \sin(t))\).

(15) Compute the value of \( \int \int_S (\nabla \times \vec{F}) \cdot \vec{n} \, dS \) where \( S \) is the upper half of the ellipsoid \( 4x^2 + 9y^2 + 36z^2 = 36 \), \( z \geq 0 \), with upward pointing normal, and \( \vec{F} = (y, x^2, (x^2 + y^2)^{3/2} e^{xyz}) \).

Solution: The presence of nasty stuff like \( e^{xyz} \) inspires us to reformulate the computation using Stokes’ theorem. The ellipse boundary can be parameterized as \( x = 3 \cos t, y = 2 \sin t, z = 0 \). The corresponding line integral is then

\[ \oint_C \vec{F} \cdot d\vec{r} = \int_0^{2\pi} \left(2 \sin t, 9 \cos^2 t, (9 \cos^2(t) + 4 \sin^2(t))^{3/2} \right) \cdot \left(-3 \sin t, 2 \cos t, 0 \right) dt = \]

\[ \int_0^{2\pi} -6 \sin^2(t) + 18 \cos^3(t) \, dt = -6\pi. \]

(16) Let \( \vec{r}(t) \) be a curve in space with unit tangent, normal, and binormal vectors \( \vec{T}, \vec{N}, \) and \( \vec{B} \). Show that \( \frac{d\vec{B}}{dt} \) is perpendicular to \( \vec{T} \).

Solution: Since \( \vec{B} = \vec{T} \times \vec{N} \), \( \frac{d\vec{B}}{dt} = \frac{d\vec{T}}{dt} \times \vec{N} + \vec{T} \times \frac{d\vec{N}}{dt} \). However \( \vec{N} \) is parallel to \( \frac{d\vec{T}}{dt} \), so in fact \( \frac{d\vec{B}}{dt} = \vec{T} \times \frac{d\vec{N}}{dt} \). Finally, note that \( \vec{a} \) is always perpendicular to \( \vec{a} \times \vec{b} \) for any vectors \( \vec{a}, \vec{b} \).

(17) Compute the flux integral \( \int \int_S \vec{F} \cdot \vec{n} \, dS \) where \( S \) is the graph of \( z = 1 - x^2 - y^2 \), with upward normal, for \( z \geq 0 \), and with \( \vec{F} = (xz, yz, 2z^2) \).

Solution: The given flux integral can be computed directly or by using the divergence theorem. I will do it both ways for comparison (on a test, one method would be sufficient). Directly: since the surface is given as a graph \( z = f(x, y) \), \( \vec{n} \, dS = d\vec{S} = (-f_x, -f_y, 1) = (2x, 2y, 1) \). Then

\[ \int \int_S \vec{F} \cdot \vec{n} \, dS = \int \int_S (xz, yz, 2z^2) \big|_{z = 2x, 2y, 1} \, dx \, dy \]
\[
\int \int_S \left(1 - x^2 - y^2\right) (2x^2 + 2y^2) + 2(1 - x^2 - y^2)^2 \, dxdy = 2 \int_0^{2\pi} \int_0^1 (1 - r^2)(2r^2 + 2(1 - r^2)) r \, drd\theta \\
= 4 \int_0^{2\pi} \int_0^1 (r - r^3) \, drd\theta = \pi.
\]

We cannot immediately apply the divergence theorem because this surface is not closed. However, we could consider the closed surface \( S_2 = S \cup S_1 \), where \( S_1 \) is the unit disk \( z = 0, x^2 + y^2 \leq 1 \) and with normal \((0, 0, -1)\). Since \( \vec{F} = 0 \) on \( S_1 \), this addition doesn’t actually affect the flux integral, i.e. \( \int \int_{S_2} \vec{F} \cdot \vec{n} \, dS_2 = \int \int_S \vec{F} \cdot \vec{n} \, dS_2 = \int \int \int R \text{div} \vec{F} \, dV \). Since \( \text{div} \vec{F} = 6z \), the value we are after can be computed in cylindrical coordinates as:

\[
\int_0^{2\pi} \int_0^1 \int_0^{1-r^2} 6z r \, dz \, d\theta \, dr = 3 \int_0^{2\pi} \int_0^1 \int_0^{1-r^2} (1 - r^2)^2 r \, drd\theta = \pi.
\]

(18) Use the divergence theorem to compute the flux of \( \vec{F} = (z^5 + x, \cos(xz), z^2) \) through the surface bounded by \( z = 0 \) and \( z = 1 - x^2 - y^2 \).

Solution: The divergence of \( \vec{F} \) is \( 1 + 2z \), so

\[
\int \int \nabla \times \vec{F} \cdot d\vec{S} = \int \int \int_R (1 + 2z) dV
\]

where \( R \) is the interior of \( S \). To evaluate this it is easiest to use cylindrical coordinates:

\[
\int_0^1 \int_0^{2\pi} (1 + 2z) r \, d\theta \, dr = \int_0^1 \int_0^{2\pi} (2r - 3r^3 + r^5) \, d\theta \, dr = 5/12 \int_0^{2\pi} d\theta = 5\pi / 6
\]