Dynamical Systems for Principal Singular Subspace Analysis

Mohammed A. Hasan
Department of Electrical & Computer Engineering
University of Minnesota Duluth
E.mail:mhasan@d.umn.edu

Abstract

The computation of the principal subspaces is an essential task in many signal processing and control applications. In this paper novel dynamical systems for finding the principal singular subspace and/or components of arbitrary matrix are developed. The proposed dynamical systems are gradient flows or weighted gradient flows derived from the optimization of certain objective functions over orthogonal constraints. Global asymptotic stability analysis and domains of attractions of these systems are examined via Liapunov theory and LaSalle invariance principle. Weighted versions of these methods for computing principal singular components are also given. Qualitative properties of the proposed systems are analyzed in detail.

Keywords: SVD, gradient flow, asymptotic stability, principal singular flow, Stiefel manifold, global convergence, constrained optimization, invariant set, LaSalle invariance principle

1 Introduction

Many signal processing tasks can efficiently be achieved by singular value decomposition (SVD) of a rectangular data matrix. The SVD can also be used for the computation of the principal components and/or subspaces. The motivation for studying dynamical systems for solving the SVD problem is to provide both discrete and analog neural systems for solving computational problems in real-time. Additionally, understanding the properties and features of such dynamical systems is helpful in determining domains of attractions and invariant sets of many principal singular subspace (PSS) dynamical systems.

There are many adaptive methods in the literature to obtain SVD of a rectangular matrix. SVD dynamical systems are developed in [1]-[11]. Algorithms for computing smallest singular triplets are proposed in [12]. Generalization of Oja’s algorithm for obtaining the SVD of a rectangular matrix is considered in [13, 14]. Cross-correlation neural networks for extracting the cross-correlation features between two high-dimensional data streams are developed in [15]-[17].

There are a number of methods for extracting principal or the minor subspaces of a positive definite matrix, however, there appears to be fewer algorithms for PSS. In this paper, several dynamical systems for computing PSS are derived and analyzed. Some of these algorithms may be considered as generalizations of principal components flows. The proposed dynamical systems converge to individual singular vectors by incorporating a diagonal matrix. Additionally, these systems are stable and self-normalized.

The following notation will be used throughout. The notation \( \mathbb{R} \) and \( \mathbb{N} \) denote the set of real numbers, and the set of positive integers, respectively. The transpose of a real matrix is denoted by \( x^T \), and the derivative of \( x \) with respect to time is written as \( x' \). If \( B \) is a square matrix, then \( \text{tr}(B) \) denotes the trace of \( B \). The identity matrix of appropriate dimension is expressed with the symbol \( I \). Finally, the derivative of \( V(x,y) \) with respect to time is denoted by \( \dot{V} \).

2 Preliminary Results

We introduce here several known results from stability theory of dynamical systems, matrix theory, and optimization over Stiefel manifolds.

2.1 Stability of Dynamical Systems

Liapunov’s direct method (also called the second method of Liapunov) is a useful tool in determining the stability of a system without explicitly integrating the associated differential equation. The method is a generalization of the idea that if there is some "measure of energy" in a system, then we can study the rate of change of the energy of the system to gain information regarding stability.

Definition (Invariant Set). Let \( g(x) : \mathbb{R}^{n \times p} \rightarrow \mathbb{R}^{n \times p} \), \( p \leq n \), be a continuously differentiable function and consider the dynamical system

\[
x' = g(x).
\]

A set \( S \) is an invariant set for the system (1) if every trajectory \( x(t) \) which starts from a point in \( S \) remains in \( S \) for all time. For example, any equilibrium point is an invariant set. The domain of attraction of an equilibrium point is also an invariant set.

We state next a few stability results for nonlinear autonomous systems. The invariant set theorems reflect the intuition that the decrease of a Liapunov function \( V \) has to gradually vanish. In other words \( \dot{V} \) has to converge to zero because \( V \) is lower bounded. Proofs of the results below can be found in [18,19].

Theorem 1 (Local Invariant Set Theorem). Consider the autonomous system (1) with \( g \) continuous and let \( V(x) : \mathbb{R}^n \rightarrow \mathbb{R} \) be a scalar function with continuous first partial derivatives. Assume that

1. for some \( l > 0 \), the set \( \Omega_l \) defined by \( V(x) \leq l \) is bounded,
2. \( V'(x) \leq 0 \) for all \( x \) in \( \Omega_l \).
Let $R$ be the set of all points within $\Omega_1$ where $V'(x) = 0$ and $M$ be the largest invariant set in $R$. Then, every solution $x(t)$ originating in $\Omega_1$ tends to $M$ as $t \to \infty$.

**Proof.** See Slotine and Li (1991) [18].

In Theorem 1, the word largest means that $M$ is the union of all invariant sets within $R$. Notice that $R$ is not necessarily connected, nor is the set $M$.

Now we are ready to state the following important theorem.

**Theorem 2 (LaSalle’s Principle for Asymptotic Stability).** Let $V(x) : \mathbb{R}^n \to \mathbb{R}$ be such that on the set $\Omega_1 = \{x \in \mathbb{R}^n : V(x) \leq l\}$ we have $V'(x) \leq 0$. Define $R = \{x \in \mathbb{R}^n : V'(x) = 0\}$. Then, if $R$ contains no other trajectories other than $x = 0$, then the origin 0 is asymptotically stable.

**Proof.** It follows directly from Theorem 1.

If $l$ in Theorem 2 extends to the whole space $\mathbb{R}^n$, then global asymptotic stability can be established.

Other versions of stability and asymptotically stability are stated next, where the neighborhood around the equilibrium point $\bar{x}$ is not directly related to the Liapunov function.

**Theorem 3 (Stability).** A solution $\bar{x}$ of $x' = g(x)$ is stable if, in a small neighborhood $\Omega$, around $\bar{x}$, there exists a positive definite function $V(x)$ such that its derivative $V(x)$ is negative semi-definite in $\Omega$.

We state next two well known results about Lagrange and Liapunov stability.

**Theorem 5 (A Lagrange Stability Theorem).** Let $W$ be a bounded neighborhood of the origin and let $W^c$ be its complement ($W^c$ is the set of all points outside $W$). Assume that $V(x)$ is a scalar function with continuous first partial derivatives in $W^c$ and satisfying:

1. $V(x) > 0$ for all $x \in W^c$,
2. $\dot{V}(x) \leq 0$ for all $x \in W^c$,
3. $V(x) \to \infty$ as $||x|| \to \infty$.

Then each solution of (1) is bounded for all $t > 0$.

The following theorem gives conditions under which we can draw conclusions about the local stability of an equilibrium point of a nonlinear system by investigating the stability of a linearized system.

**Theorem 6 (Liapunov’s Indirect Method).** Let $x = 0$ be an equilibrium point for the nonlinear system $\dot{x} = g(x)$, where $g : D \to \mathbb{R}^n$ is continuously differentiable and $D$ is a neighborhood of the origin. Let the Jacobian matrix $A$ at $x = 0$ be:

$$A = \frac{\partial g}{\partial x}_{|x=0}.$$  \hspace{1cm} (2)

Let $\lambda_i$, $i = 1, \ldots, n$ be the eigenvalues of $A$. Then:

1. The origin is asymptotically stable if $Re(\lambda_i) < 0$ for all eigenvalues of $A$.
2. The origin is unstable if $Re(\lambda_i) > 0$ for any of the eigenvalues of $A$.

Here $Re(\lambda)$ denotes the real part of $\lambda$.

**Proof.** The proof of this theorem can be found in Khalil (2002) [19].

If some of the eigenvalues of $\frac{\partial g}{\partial x}_{|x=0}$ lie on the imaginary axis, further analysis using the central manifold theory is needed to establish stability.

### 2.2 Matrix Analysis

In this section, we introduce several results from matrix theory. These results will be used later to prove qualitative properties about the behavior of solutions of dynamical systems as $t \to \infty$.

**Proposition 7.** Let $A \in \mathbb{R}^{n \times n}$ and $x \in \mathbb{R}^{n \times p}$, $n, p \in \mathbb{N}$. If $A + A^T$ is positive definite then $A$, $A^T$, $x^T A x$, and $x^T A^T x$ are nonsingular for any full rank matrix $x$.

**Proof.** Assume that $A + A^T = B$ is positive definite and that $A$ is singular so the $B y = 0$ for some non-zero vector $y$. Then $y^T A y + y^T A^T y = y^T B y = 0$. This is a contradiction since $B$ is positive definite. Therefore $A$ is nonsingular. Similarly, for any given full rank matrix $x$, $x^T A x + x^T A^T x = x^T B x$ is positive definite and hence $x^T A x$ is nonsingular.

**Proposition 8.** Let $B \in \mathbb{R}^{n \times n}$ and assume that all eigenvalues of $B^2$ are distinct. If $B^2 = (B^T)^2$, then $B^T = B$.

**Proof.** Assume that $B = W D_2 W^{-1}$ and $B^2 = Z D_2 Z^{-1}$ are eigendecompositions of $B$ and $B^T$, respectively, where $D_1$ and $D_2$ are diagonal having distinct eigenvalues. Then $(B^T)^2 = B^2$ implies that $W D_2 W^{-1} = Z D_2 Z^{-1}$. Consequently, $Z^{-1} W D_2 = D_2 Z^{-1} W$. Since it is assumed that all eigenvalues of $B$ (or equivalently $D_2$) are distinct, it follows that $Z^{-1} W = D_3$ for some diagonal matrix $D_3$. This shows that $W = Z D_3$ and $B = W D_2 W^{-1} = Z D_3 D_2 Z^{-1} = B^T$, i.e., $B$ is symmetric.

**Proposition 9** [20]. Let $B, D \in \mathbb{R}^{p \times p}$ and assume that $D$ is diagonal and all eigenvalues of $D$ are distinct.

1. If $B^2 = DB$ is diagonal, then $B$ is diagonal.
2. If $BD = DB$, then $B$ is diagonal.

**Proposition 10.** Let $B \in \mathbb{R}^{n \times n}$ and assume that all eigenvalues of $B^2$ are distinct. If $B = P B^T$, where $P$ is symmetric, then $B^2 = (B^T)^2$. The rest of the conclusion follows from Proposition 8.

**Remark 1.** In the above proof one can show that $B = P^k B P^k$ for every integer $k$. This suggests that the eigenvalues of $P$ are all $1$ or $-1$.

**Proposition 11.** Let $B \in \mathbb{R}^{n \times n}$ and assume that $B + B^T$ is positive definite. If $B = P B^T$, where $P$ is symmetric, then $B^T = B$ and $P = I$.

**Proposition 12.** Let $B \in \mathbb{R}^{n \times n}$ and assume that all eigenvalues of $B$ are distinct. If $B D = DB$ and $B^2 D = Q B$ where $D$ is diagonal matrix with distinct eigenvalues, $P$ and $Q$ are symmetric matrices, then $B^T = B$ and $D = P$. Moreover, if $B$ is nonsingular, then $P = Q = D$.

**Proof.** Since $B^2 D = B P B^T$ and $(B^T)^2 D = B^T P B$ are symmetric, it follows that $(B^2 + (B^T)^2) D$ is symmetric. This implies that $B^2 + (B^T)^2$ is diagonal. Assume that $B^2 + (B^T)^2 = D_1$ for some diagonal matrix $D_1$. Since $B^2 D = D_1$ is symmetric, we have $B^2 D = D(1)^2 = D(D_1 - B^2)$ or $B^2 D + DB^2 = DD_1$. Hence $B^2$ is diagonal from which it follows that $B$ is diagonal. Hence $P = Q = D$ provided that $B$ is nonsingular.

**Proposition 13.** Let $B \in \mathbb{R}^{n \times n}$ and assume that all eigenvalues of $B^T B$ are distinct. Assume that $B^T B = D_1$ and $B^T B^2 = D_2$, where $D_1, i = 1, 2$, are diagonal matrices. Then...
there is a permutation matrix \( J \) such that \( B = JD_1^T \) and \( D_2 = JD_2, F = JD_1 D_3 \), i.e., \( B = JD_1 D_3 \) where \( D_3 = I \).

**Proof.** There are two orthogonal matrices \( U_1 \) and \( U_2 \) such that
\[
B = U_1 D_1 T U_2 \quad \text{and} \quad B = D_2 U_2 U_1 T.
\]
Clearly, \( U_1 D_1 U_2 T = U_1 U_2 D_1 U_2 T = U_2 D_1 U_2 T \).
From these equations, it follows that
\[
U_1 D_1 U_2 D_2 = D_1 U_2 U_1 T.
\]
Proposition 9 implies that \( U_2 D_1 U_2 D_2 = D_1 U_2 D_2, D_2 \) for some diagonal matrix \( D_2 \). The equation \( B = U_1 D_1 U_2 D_2 \) yields that \( D_2 = I \). Therefore, \( B = U_1 D_1 U_2 D_2 \) or \( U_2 D_1 U_2 D_2 = D_1 U_2 D_2, \) Hence \( U_2 = J \) where \( J \) is a permutation matrix, i.e., \( B = JD_1 D_2 \) and \( B = (JD_1, J) = D_2, J = D_2, J \) where \( J D_2 J = D_2 \).

### 2.3 Optimization Over a Stiefel Manifold

Let \( f(x): \mathbb{R}^{n \times P} \rightarrow \mathbb{R}, p \leq n \), be continuously differentiable function and consider the constrained optimization problem
\[
\text{Maximize } f(x) \text{ subject to } x \in Z, \quad (3)
\]
where \( Z = \{ x \in \mathbb{R}^{n \times P} : x^T x = I \} \) is a Stiefel manifold. It is known [21] that the gradient of \( f \) with respect to the Stiefel manifold \( Z \) is
\[
\nabla_N f(x) = \nabla f(x) - x (\nabla f(x))^T x. \quad (4a)
\]
Note that \( \nabla_N f \) is sometimes called the natural gradient of \( f \) over \( Z \). A necessary condition for \( x^* \) to be a solution for (3) is that \( \nabla_N f(x^*) = 0 \). Under mild conditions, the dynamical system
\[
x' = \nabla_N f(x) = \nabla f(x) - x (\nabla f(x))^T x, \quad (4b)
\]
converges to a maximizer of \( f \) over \( Z \).

The following results provides stability analysis for dynamical system (4b) for maximizing a function over a Stiefel manifold.

**Theorem 14.** Let \( f \) be as described above and assume that \( x^T \nabla f(x) + (\nabla f(x))^T x \) is positive definite for every full rank matrix \( x \) and consider the dynamical system (4b) with the initial condition \( x(0) = x_0 \). Then (4b) is globally asymptotically stable.

**Proof.** Let \( V(x) = \frac{1}{2} \text{tr}(x^T x - I)^2 \), where \( I \) denotes a \( p \times p \) identity matrix. Then
\[
V'(x) = -\text{tr}((x^T x - I) (x^T \nabla f(x) + (\nabla f(x))^T x) (x^T x - I)) \leq 0.
\]
Since \( x^T \nabla f(x) + (\nabla f(x))^T x \) is positive definite, then \( V'(x) = 0 \) if and only if \( x^T x - I = 0 \). Thus the largest invariant set \( M \) is given by \( M = \{ x \in \mathbb{R}^{n \times P} : x^T x = I \} \). Then, every solution \( x(t) \) originating any where in \( \mathbb{R}^{n \times P} \) (since \( \Omega \in \mathbb{R}^{n \times P} \)) tends to \( M \) as \( t \to \infty \). Note that for every \( x \in M \) we have \( x^T x = x^T x \), i.e., \( x^T x \) is a projection. If \( x \) is full rank, then \( x^T x \) is \( x \). This established that the dynamical system (4b) is globally asymptotically stable. The rest of the conclusion follows from Proposition 10.

**Corollary 15.** Consider the dynamical system (4b) and assume that \( x(t) \to x \) as \( t \to \infty \). If all eigenvalues of \( (\nabla f(x))^T x \) are distinct, then \( (\nabla f(x))^T x \) is symmetric. Additionally, if \( (\nabla f(x))^T x \) is invertible then \( x^T x = I \).

**Proof.** Let \( B = (\nabla f(x))^T x \), then \( B^T = PB \), where \( P = x^T x \). Clearly, \( (B^T)^2 = B^T PB \) is symmetric and therefore \( (B^T)^2 = B^2 \). Since all eigenvalues of \( B^2 \) are distinct, it follows from Proposition 8 that \( B \) is symmetric and \( (1 - P)B = 0 \). Thus, \( P = I \) if \( B \) is invertible.

The next result provides stability analysis for dynamical systems for minimizing a function over a Stiefel manifold.

### 3 Principal Singular Component Analysis (PSCA)

Several dynamical systems are derived for computing principal singular components of a rectangular matrix \( A \). Different versions can be obtained from solving constrained optimization problems using the Lagrange multiplier theory directly [22] or by applying the results of Section 2.3.

#### 3.1 Dynamical System I

By applying Theorem 5 for maximizing \( \text{tr}(x^T A P) \) over the double Stiefel manifolds defined by the constraints; \( x^T x = I \), and \( y^T y = I \), we obtain the following system:
\[
x' = AyD - x D_1^T A^T x, \quad (8)
\]
\[
y' = A^T x D - y D_2^T A. \quad (8)
\]
Here $D$ is positive definite diagonal matrix with distinct eigenvalues. Dynamical systems that are similar to (8) have been derived and analyzed in [4,9] for the case $D = I$. In this paper, we will analyze the qualitative behavior of this system by examining global stability. Stability can be established via the Liapunov function $V(x, y) = \frac{1}{2} \text{tr}((x^T x - I)^2 + (y^T y - I)^2)$ defined over the open set $\Omega$ where

$$\Omega = \{ x \in \mathbb{R}^{n \times p}, y \in \mathbb{R}^{m \times p} : x^T A y D + D y^T A^T x \text{ is positive definite} \}.$$  

Note that if $(x, y) \in \Omega$, then each of $x$ and $y$ are full rank. Also, if $x$ and $y$ consist of the left and right singular vectors corresponding to the largest $p$ singular values, then $(x, y) \in \Omega$, i.e., each full rank equilibrium solution of (8) corresponds to nonzero singular values of $A$ belongs to $\Omega$.

The time derivative of $V$ along the trajectories $x(t)$ and $y(t)$ is

$$\dot{V} = \text{tr}\{(x^T x - I)\{x^T A y D - x^T x D y^T A^T x + D y^T A^T x \} - x^T A y D x^T x + (y^T y - I)\{y^T A^T x D - y^T y D x^T A^T x \} + D x^T A y - y^T A^T x D y^T y\}$$

$$= \text{tr}\{(x^T x - I)(x^T A y D + D y^T A^T x)(I - x^T x) + (y^T y - I)(x^T A y D + D y^T A^T x)(I - y^T y)\} \leq 0.$$

We also note that any solution of $\dot{V} = 0$ within $\Omega$ yields that $x^T x = I$ and $y^T y = I$. Thus the largest invariant set $M$ within $\Omega$ is given by $M = \{ x \in \mathbb{R}^{n \times p}, y \in \mathbb{R}^{m \times p} : x^T x = I, y^T y = I \}$. This means that every solution $(x(t), y(t))$ originating at any $(x(0), y(0)) \in \Omega$ tends to $M$ as $t \to \infty$. This indicates that (8) is asymptotically stable over $\Omega$.

Next, we outline a proof showing that $x(t)$ and $y(t)$ converge to the actual principal left and right singular vectors of $A$ as $t \to \infty$. Let $x(t) \to x$ as $t \to \infty$ and set $B = x^T A y D$, $P = x^T x$ and $Q = y^T y$, then $BD = PDB^T$ and $B^T D = QDB$. The last two equations imply that $(BD)^2$ and $(B^T D)^2$ are symmetric. If $D$ is chosen so that all eigenvalues of $(BD)^2$ are distinct, then by Proposition 8 $BD$ and $B^T D$ are symmetric. Therefore, $(B + B^T D) = D(B + B^T)$ from which it follows that $B + B^T$ is diagonal (Proposition 9). Assume that $B + B^T = D_1$ for some diagonal matrix $D_1$. Since $B^T D$ is symmetric, we have $B^T D = DB$ and $BD + B^T D = BD + DB = DD_1$. Hence $B$ is diagonal from which it follows that $P = I$ and $Q = I$.

### 3.2 Dynamical System II

This system is derived from System I via the change of variables $z = xD^{-1}$ and $w = yD^{-1}$ and then replacing $z$ and $w$ with $x$ and $y$, respectively:

$$x' = AyD - xy^T A^T x,$$

$$y' = A^T xD - yx^T Ay.$$  

As in System I, stability analysis of (10) can be established via the Liapunov function $V(x, y) = \frac{1}{2} \text{tr}((x^T x - D)^2 + (y^T y - D)^2)$ defined over the open set $\Omega$, where

$$\Omega = \{ x \in \mathbb{R}^{n \times p}, y \in \mathbb{R}^{m \times p} : x^T A y + y^T A^T x \text{ is positive definite} \}.$$  

Note that if $(x, y) \in \Omega$, then each of $x$ and $y$ are full rank. Also, if $x$ and $y$ consist of the left and right singular vectors corresponding to $p$ nonzero singular values, respectively, then $(x, y) \in \Omega$.

The time derivative of $V$ along the trajectories $x(t)$ and $y(t)$ simplifies to

$$\dot{V} = \text{tr}\{(x^T x - D)(x^T Ay + y^T A^T x)(D - x^T x) + (y^T y - D)(x^T Ay + y^T A^T x)(D - y^T y)\} \leq 0.$$  

We also note that any solution of $\dot{V} = 0$ within $\Omega$ yields that $x^T x = D$ and $y^T y = D$. Thus the largest invariant set $M$ within $\Omega$ is given by $M = \{ x \in \mathbb{R}^{n \times p}, y \in \mathbb{R}^{m \times p} : x^T x = D, y^T y = D \}$. This means that every solution $(x(t), y(t))$ originating at any $(x(0), y(0)) \in \Omega$ tends to $M$ as $t \to \infty$. This indicates that (10) is stable over $\Omega$.

It can be shown that the matrices $x(t)^T x(t)$, $x(t)^T Ay(t)$, and $y(t)^T y(t)$ converge to diagonal matrices as $t \to \infty$ provided that all the diagonal elements of the diagonal matrix $D$ are distinct. This follows directly from Proposition 12.

**Remark 3:** PSAC dynamical systems may be weighed differently. For example, each of the following systems can be shown to converge to PSAC:

$$x' = AyD - xy^T A^T x,$$

$$y' = A^T xD - yx^T Ay,$$

$$x' = Ay - xy^T A^T x,$$

$$y' = A^T xD - yx^T Ay,$$

$$x' = Ay - xy^T A^T xD,$$

$$y' = A^T xD - yx^T Ay,$$

$$x' = Ay - xy^T A^T x,$$

$$y' = A^T xD - yx^T Ay,$$

$$x' = Ay - xy^T A^T xD,$$

$$y' = A^T xD - yx^T Ay.$$  

Stability analysis of each of the above systems can be established as in Systems I and II via a Liapunov function of the form $V(x, y) = \frac{1}{2} \text{tr}((x^T x - D_1)^2 + (y^T y - D_2)^2)$. Depending on the system to be analyzed, $D_1 = D$ or $D_1 = I$.

### 3.3 Dynamical System III

This system is obtained from solving the optimization problem

$$\text{Maximize } \text{tr}(x^T Ay) \text{ subject to } x^T x = D, y^T y = D.$$  

Using the method of Lagrange multipliers yields the following system:

$$x' = Ay - x(Dx^T Ay + y^T A^T xD),$$

$$y' = A^T xD - y(Dx^T Ay + y^T A^T xD).$$  

**Theorem 17:** In system (18), if $x(t) \to x$ and $y(t) \to y$ as $t \to \infty$, then $x^T Ay$ is diagonal and $x^T x = y^T y = I$, provided that $x^T A y$ and $D$ have distinct eigenvalues.

**Proof:** Let $B = x^T A y$, $P = x^T x$ and $Q = y^T y$, then $B = PS$ and $B^T = QS$, where $S = DB + B^T D$. The last two equations imply that $(P + Q)S = S(P + Q)$ since and $B + B^T$ is symmetric. Since $P + Q$ and $S$ are symmetric and commute, it follows that if $P + Q$ is expressed as $P + Q = WD_2W^T$, where $D_2$ is a diagonal matrix and $W$ is orthogonal, then $S = W D_2 W^T$ for
some diagonal matrix $D_3$. The equations $B = PS$ and $B^T = QS$ imply that $SP = QS$ and therefore,
\[ WD_3W^T P = (WD_4W^T - P)WD_4W^T. \]
Equivalently,
\[ D_3W^T PW + W^TPWD_4 = D_2. \]
Therefore, $W^T PW$ is diagonal. Assume that $W^T PW = D_4$ for some diagonal matrix $D_4$. This shows that $P = WD_4W^T$ and hence $Q = W(D_3 - D_1)W^T$. Since the matrices $P, Q, S$ have the same set of eigenvectors, they all commute. Thus $B = PS = SP = B^T$, i.e., $B$ is symmetric. Again, the equations $B = PS$ and $B = B^T = QS$ imply that $P = Q = \frac{1}{2}WD_2W^T$. Also $B = \frac{1}{2}WD_2D_3W^T$. Thus the equation $B = PS$ yields
\[ \frac{1}{2}W D_2 D_3 W^T = \frac{1}{2}W D_2 W^T (D_2 D_3 W^T + \frac{1}{2}W D_2 D_3 W^T D) \]
from which we have
\[ 2D_3 = W^T DW D_2 D_3 + D_2 D_3 W^T DW. \]
Hence $W^T DW$ is diagonal which means that $W = J$ for some permutation matrix $J$. The equation $B + B^T = (P + Q)S = (P + Q)(B + B^T)$ implies that $2P D = QD = I$ or $P = Q = \frac{1}{2}I$. It follows that $B, P, Q$ are all diagonal.

4 Conclusion

New principal singular subspace and component flows (3) are derived and analysed. Liapunov and Lagrange stability of some of these flows are examined. Different dynamical systems were obtained by weighting a given system with a diagonal matrix. These flows are new and are derived using constrained and unconstrained optimization methods. Some of the proposed rules generalize Oja’s principal component flow and other known flows for singular value decomposition. Further analysis is needed to explore numerical stability and convergence. Extension of the proposed rules to complex data and matrices can be achieved with minor modifications.

References
