An Extension of Theorems of Hechner and Heinkel

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1 Introduction and Main Result

It is a great pleasure for us to contribute this paper in honour of Professor Miklós Csörgő's work on the occasion of his 80th birthday.

Throughout, let $(\mathbf{B}, \|\cdot\|)$ be a real separable Banach space equipped with its Borel σ-algebra $\mathscr{B}$ (= the σ-algebra generated by the class of open subsets of $\mathbf{B}$ determined by $\|\cdot\|$) and let $\{X_n;\ n \geq 1\}$ be a sequence of independent copies of a $\mathbf{B}$-valued random variable X defined on a probability space $(\Omega, \mathscr{F}, \mathbb{P})$. As usual, let $S_n = \sum_{k=1}^{n} X_k,\ n \geq 1$ denote their partial sums. If $0 < p < 2$ and if X is a real-valued random variable (that is, if $\mathbf{B} = \mathbb{R}$), then

$$\lim_{n\to\infty} \frac{S_n}{n^{1/p}} = 0 \text{ almost surely (a.s.)}$$

if and only if

$$\mathbb{E}|X|^p < \infty \text{ where } \mathbb{E}X = 0 \text{ whenever } p \geq 1.$$

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This is the celebrated Kolmogoroff-Marcinkiewicz-Zygmund strong law of large numbers (SLLN); see Kolmogoroff [9] for $p = 1$ and Marcinkiewicz and Zygmund [14] for $p \neq 1$.

The classical Kolmogoroff SLLN in real separable Banach spaces was established by Mourier [15]. The extension of the Kolmogoroff-Marcinkiewicz-Zygmund SLLN to **B**-valued random variables is independently due to Azlarov and Volodin [1] and de Acosta [4].

Theorem 1 (Azlarov and Volodin [1] and de Acosta [4]). *Let $0 < p < 2$ and let $\{X_n;\ n \geq 1\}$ be a sequence of independent copies of a* **B***-valued random variable X. Then*

$$\lim_{n\to\infty} \frac{S_n}{n^{1/p}} = 0 \ \ a.s.$$

if and only if

$$\mathbb{E}\|X\|^p < \infty \ \ and \ \ \frac{S_n}{n^{1/p}} \to_{\mathbb{P}} 0.$$

Let $0 < p \leq 2$ and let $\{\Theta_n;\ n \geq 1\}$ be a sequence of i.i.d. stable random variables each with characteristic function $\psi(t) = \exp\{-|t|^p\},\ -\infty < t < \infty$. Then **B** is said to be of *stable type p* if $\sum_{n=1}^{\infty} \Theta_n v_n$ converges a.s. whenever $\{v_n :\ n \geq 1\} \subseteq \mathbf{B}$ with $\sum_{n=1}^{\infty} \|v_n\|^p < \infty$. Equivalent characterizations of a Banach space being of stable type p and properties of stable type p Banach spaces may be found in Ledoux and Talagrand [10]. Some of these properties are summarized in Li, Qi, and Rosalsky [12].

At the origin of the current investigation is the following recent and striking result by Hechner [6] for $p = 1$ and Hechner and Heinkel [7, Theorem 5] for $1 < p < 2$ which are new even in the case where the Banach space **B** is the real line. The earliest investigation that we are aware of concerning the convergence of the series $\sum_{n=1}^{\infty} \frac{1}{n}\left(\frac{\mathbb{E}|S_n|}{n}\right)$ was carried out by Hechner [5] for the case where $\{X_n;\ n \geq 1\}$ is a sequence of i.i.d. mean zero real-valued random variables.

Theorem 2 (Hechner [6, Theorem 2.4.1] for $p = 1$ and Hechner and Heinkel [7, Theorem 5] for $1 < p < 2$). *Suppose that* **B** *is of stable type p for some $p \in [1, 2)$ and let $\{X_n;\ n \geq 1\}$ be a sequence of independent copies of a* **B***-valued variable X with $\mathbb{E}X = 0$. Then*

$$\sum_{n=1}^{\infty} \frac{1}{n}\left(\frac{\mathbb{E}\|S_n\|}{n^{1/p}}\right) < \infty$$

if and only if

$$\begin{cases} \mathbb{E}\|X\| \ln(1 + \|X\|) < \infty & if\ p = 1, \\ \displaystyle\int_0^{\infty} \mathbb{P}^{1/p}\left(\|X\| > t\right) dt < \infty & if\ 1 < p < 2. \end{cases}$$

Inspired by the above discovery by Hechner [6] and Hechner and Heinkel [7], Li, Qi, and Rosalsky [12] obtained sets of necessary and sufficient conditions for

$$\sum_{n=1}^{\infty} \frac{1}{n} \left(\frac{\|S_n\|}{n^{1/p}} \right) < \infty \text{ a.s.}$$

for the three cases: $0 < p < 1$, $p = 1$, $1 < p < 2$ (see Theorem 2.4, Theorem 2.3, and Corollary 2.1, respectively of Li, Qi, and Rosalsky [12]). Again, these results are new when $\mathbf{B} = \mathbb{R}$; see Theorem 2.5 of Li, Qi, and Rosalsky [12]. Moreover for $1 \le p < 2$, Li, Qi, and Rosalsky [12, Theorems 2.1 and 2.2] obtained necessary and sufficient conditions for

$$\sum_{n=1}^{\infty} \frac{1}{n} \left(\frac{\mathbb{E}\|S_n\|}{n^{1/p}} \right) < \infty$$

for general separable Banach spaces.

This paper is devoted to an extension of Theorem 2 above and Theorems 2.1 and 2.2 of Li, Qi, and Rosalsky [12]. More specifically, the main result of this paper is the following theorem. We note that no conditions are being imposed on the Banach space $\mathbf{B}$.

Theorem 3. *Let $0 < p < 2$ and $0 < q < \infty$. Let $\{X_n;\ n \ge 1\}$ be a sequence of independent copies of a $\mathbf{B}$-valued random variable X. Then*

$$\sum_{n=1}^{\infty} \frac{1}{n} \mathbb{E} \left(\frac{\|S_n\|}{n^{1/p}} \right)^q < \infty \tag{1}$$

if and only if

$$\sum_{n=1}^{\infty} \frac{1}{n} \left(\frac{\|S_n\|}{n^{1/p}} \right)^q < \infty \ \ a.s. \tag{2}$$

and

$$\begin{cases} \displaystyle\int_0^{\infty} \mathbb{P}^{q/p} \left(\|X\|^q > t \right) dt < \infty & \text{if } 0 < q < p, \\ \mathbb{E}\|X\|^p \ln(1 + \|X\|) < \infty & \text{if } q = p, \\ \mathbb{E}\|X\|^q < \infty & \text{if } q > p. \end{cases} \tag{3}$$

Furthermore, each of (1) and (2) implies that

$$\lim_{n\to\infty} \frac{S_n}{n^{1/p}} = 0 \ \ a.s. \tag{4}$$

For $0 < q < p$, (1) and (2) are equivalent so that each of them implies that (3) and (4) hold.

Remark 1. Let $q = 1$. Then one can easily see that Theorems 2.1 and 2.2 of Li, Qi, and Rosalsky [12] follow from Theorem 3 above.

Remark 2. It follows from the conclusion (4) of Theorem 3 that, if (2) holds for some $q = q_1 > 0$ then (2) holds for all $q > q_1$.

The proof of Theorem 3 will be given in Sect. 3. For proving Theorem 3, we employ new versions of the classical Lévy [11], Ottaviani [3, p. 75], and Hoffmann-Jørgensen [8] inequalities which have recently been obtained by Li and Rosalsky [13] (stated in Sect. 2). As an application of the new versions of the classical Lévy [11] and Hoffmann-Jørgensen [8] inequalities, in Theorem 7 some general results concerning sums of the form $\sum_{n=1}^{\infty} a_n \| \sum_{k=1}^{n} V_k \|^q$ (where the $a_n \geq 0$ and $\{V_k;\ k \geq 1\}$ is a sequence of independent symmetric $\mathbf{B}$-valued random variables and $q > 0$) are established; these results are key components in the proof of Theorem 3.

2 New Versions of Some Classical Stochastic Inequalities

Li and Rosalsky [13] have recently obtained new versions of the classical Lévy [11], Ottaviani [3, p. 75], and Hoffmann-Jørgensen [8] inequalities. In this section we state the results obtained by Li and Rosalsky [13] which we use for proving the main result in this paper. Then, as an application of the new versions of the classical Lévy and Hoffmann-Jørgensen [8] inequalities, we establish some general results for sums of the form $\sum_{n=1}^{\infty} a_n \| \sum_{k=1}^{n} V_k \|^q$, where the a_n are nonnegative and where $\{V_k;\ k \geq 1\}$ is a sequence of independent symmetric $\mathbf{B}$-valued random variables and $q > 0$.

Let $\{V_n;\ n \geq 1\}$ be a sequence of independent $\mathbf{B}$-valued random variables defined on a probability space $(\Omega, \mathscr{F}, \mathbb{P})$. Let $\mathbf{B}^{\infty} = \mathbf{B} \times \mathbf{B} \times \mathbf{B} \times \cdots$ and $g : \mathbf{B}^{\infty} \to \overline{\mathbb{R}}_+ = [0, \infty]$ be a measurable function. Let

$$T_n = g\left(V_1, \ldots, V_n, 0, \ldots\right), \quad Y_n = g\left(0, \ldots, 0, V_n, 0, \ldots\right), \quad M_n = \max_{1 \leq j \leq n} T_j, \quad N_n = \max_{1 \leq j \leq n} Y_j$$

for $n \geq 1$, and

$$M = \sup_{n \geq 1} T_n, \quad N = \sup_{n \geq 1} Y_n.$$

The following result, which is a new general version of Lévy's inequality, is Theorem 2.1 of Li and Rosalsky [13].

Theorem 4 (Li and Rosalsky [13]). *Let $\{V_n;\ n \geq 1\}$ be a sequence of independent symmetric $\boldsymbol{B}$-valued random variables. Let $g : \mathbf{B}^{\infty} \to \overline{\mathbb{R}}_+ = [0, \infty]$ be a measurable function such that for all $\mathbf{x}, \mathbf{y} \in \mathbf{B}^{\infty}$,*

$$g\left(\frac{\mathbf{x} + \mathbf{y}}{2}\right) \leq \alpha \max\left(g(\mathbf{x}),\ g(\mathbf{y})\right), \tag{5}$$

where $1 \leq \alpha < \infty$ is a constant, depending only on the function g. Then for all $t \geq 0$, we have

$$\mathbb{P}\left(M_n > t\right) \leq 2\mathbb{P}\left(T_n > \frac{t}{\alpha}\right)$$

and

$$\mathbb{P}\left(N_n > t\right) \leq 2\mathbb{P}\left(T_n > \frac{t}{\alpha}\right).$$

Moreover if $T_n \to T$ in law, then for all $t \geq 0$, we have

$$\mathbb{P}(M > t) \leq 2\mathbb{P}\left(T > \frac{t}{\alpha}\right)$$

and

$$\mathbb{P}(N > t) \leq 2\mathbb{P}\left(T > \frac{t}{\alpha}\right).$$

Remark 3. Theorem 4 includes the classical Lévy inequality [11] as a special case if $\mathbf{B} = \mathbb{R}$ and $g\left(x_1, x_2, \ldots, x_n, \ldots\right) = \left|\sum_{i=1}^{n} x_i\right|$, $\left(x_1, x_2, \ldots, x_n, \ldots\right) \in \mathbb{R}^{\infty}$. Theorem 4 is due to Hoffmann-Jørgensen [8] for the special case of $\alpha = 1$.

The following result, which is Theorem 2.2 of Li and Rosalsky [13], is a new general version of the classical Ottaviani [3, p. 75] inequality.

Theorem 5 (Li and Rosalsky [13]). *Let $\{V_n;\ n \geq 1\}$ be a sequence of independent $\mathbf{B}$-valued random variables. Let $g: \mathbf{B}^{\infty} \to \overline{\mathbb{R}}_{+} = [0, \infty]$ be a measurable function such that for all $\mathbf{x}, \mathbf{y} \in \mathbf{B}^{\infty}$,*

$$g(\mathbf{x} + \mathbf{y}) \leq \beta\left(g(\mathbf{x}) + g(\mathbf{y})\right), \tag{6}$$

where $1 \leq \beta < \infty$ is a constant, depending only on the function g. Then for all $n \geq 1$ and all nonnegative real numbers t and u, we have

$$\mathbb{P}\left(M_n > t + u\right) \leq \frac{\mathbb{P}\left(T_n > \frac{t}{\beta}\right)}{1 - \max_{1 \leq k \leq n-1} \mathbb{P}\left(D_{n,k} > \frac{u}{\beta}\right)},$$

where

$$D_{n,j} = g\left(0, \cdots, 0, -V_{j+1}, \cdots, -V_n, 0, \cdots\right),\ j = 1, 2, \ldots, n-1.$$

In particular, if for some $\delta \geq 0$,

$$\max_{1 \leq k \leq n-1} \mathbb{P}\left(D_{n,k} > \frac{\delta}{\beta}\right) \leq \frac{1}{2},$$

then for every $t \geq \delta$, we have

$$\mathbb{P}\left(M_n > 2t\right) \leq 2\mathbb{P}\left(T_n > \frac{t}{\beta}\right).$$

Remark 4. The classical Ottaviani inequality follows from Theorem 5 if $\mathbf{B} = \mathbb{R}$ and

$$g\left(x_1, x_2, \ldots, x_n, \ldots\right) = \left|\sum_{k=1}^{n} x_i\right|, \quad \left(x_1, x_2, \ldots, x_n, \ldots\right) \in \mathbb{R}^{\infty}.$$

The following result, which is Theorem 2.3 of Li and Rosalsky [13], is a new general version of the classical Hoffmann-Jørgensen inequality [8].

Theorem 6 (Li and Rosalsky [13]). *Let $\{V_n;\ n \geq 1\}$ be a sequence of independent symmetric $\boldsymbol{B}$-valued random variables. Let $g\ :\ \mathbf{B}^{\infty} \to \overline{\mathbb{R}}_{+} = [0, \infty]$ be a measurable function satisfying conditions (5) and (6) . Then for all nonnegative real numbers s, t, and u, we have*

$$\begin{aligned}
\mathbb{P}\left(T_n > s+t+u\right) &\leq \mathbb{P}\left(N_n > \frac{s}{\beta^2}\right) + 2\mathbb{P}\left(T_n > \frac{u}{\alpha\beta}\right)\mathbb{P}\left(M_n > \frac{t}{\beta^2}\right) \\
&\leq \mathbb{P}\left(N_n > \frac{s}{\beta^2}\right) + 4\mathbb{P}\left(T_n > \frac{u}{\alpha\beta}\right)\mathbb{P}\left(T_n > \frac{t}{\alpha\beta^2}\right),
\end{aligned}$$

$$\mathbb{P}\left(M_n > s+t+u\right) \leq 2\mathbb{P}\left(N_n > \frac{s}{\alpha\beta^2}\right) + 8\mathbb{P}\left(T_n > \frac{u}{\alpha^2\beta}\right)\mathbb{P}\left(T_n > \frac{t}{\alpha^2\beta^2}\right),$$

and

$$\mathbb{P}\left(M > s+t+u\right) \leq 2\mathbb{P}\left(N > \frac{s}{\alpha\beta^2}\right) + 4\mathbb{P}\left(M > \frac{u}{\alpha^2\beta}\right)\mathbb{P}\left(M > \frac{t}{\alpha\beta^2}\right).$$

Remark 5. The classical Hoffmann-Jørgensen inequality [8] follows from Theorem 6 if $\alpha = 1$ and $\beta = 1$.

For illustrating the new versions of the classical Lévy [11] and Hoffmann-Jørgensen [8] inequalities, i.e., Theorems 4 and 6 above, we now establish the following general result.

Theorem 7. *Let $q > 0$ and let $\{a_n;\ n \geq 1\}$ be a sequence of nonnegative real numbers such that $\sum_{n=1}^{\infty} a_n < \infty$. Let $\{V_k;\ k \geq 1\}$ be a sequence of independent symmetric $\mathbf{B}$-valued random variables. Write*

$$b_n = \sum_{k=n}^{\infty} a_k, \quad n \geq 1$$

and

$$\alpha = \begin{cases} 2^{1-q}, \ \textit{if} \ 0 < q \le 1 \\ \\ 1, \quad \ \textit{if} \ q > 1. \end{cases} \quad \textit{and} \ \beta = \begin{cases} 1, \quad \ \textit{if} \ 0 < q \le 1 \\ \\ 2^{q-1}, \ \textit{if} \ q > 1. \end{cases} \tag{7}$$

Then, for all nonnegative real numbers s, t, and u, we have that

$$\mathbb{P}\left(\sup_{n\ge 1} b_n \|V_n\|^q > t\right) \le 2\mathbb{P}\left(\sum_{n=1}^{\infty} a_n \left\|\sum_{i=1}^{n} V_i\right\|^q > \frac{t}{\alpha}\right) \tag{8}$$

and

$$\mathbb{P}\left(\sum_{n=1}^{\infty} a_n \left\|\sum_{i=1}^{n} V_i\right\|^q > s+t+u\right) \le \mathbb{P}\left(\sup_{n\ge 1} b_n \|V_n\|^q > \frac{s}{\beta^2}\right) + 4\mathbb{P}\left(\sum_{n=1}^{\infty} a_n \left\|\sum_{i=1}^{n} V_i\right\|^q > \frac{u}{\alpha\beta}\right)\mathbb{P}\left(\sum_{n=1}^{\infty} a_n \left\|\sum_{i=1}^{n} V_i\right\|^q > \frac{t}{\alpha\beta^2}\right). \tag{9}$$

Furthermore, we have that

$$\mathbb{E}\left(\sup_{n\ge 1} b_n \|V_n\|^q\right) \le 2\alpha\mathbb{E}\left(\sum_{n=1}^{\infty} a_n \left\|\sum_{i=1}^{n} V_i\right\|^q\right) \tag{10}$$

and

$$\mathbb{E}\left(\sum_{n=1}^{\infty} a_n \left\|\sum_{i=1}^{n} V_i\right\|^q\right) \le 6(\alpha+\beta)^3\mathbb{E}\left(\sup_{n\ge 1} b_n \|V_n\|^q\right) + 6(\alpha+\beta)^3 t_0, \tag{11}$$

where

$$t_0 = \inf\left\{t > 0;\ \mathbb{P}\left(\sum_{n=1}^{\infty} a_n \left\|\sum_{i=1}^{n} V_i\right\|^q > t\right) \le 24^{-1}(\alpha+\beta)^{-3}\right\}.$$

Proof. For $m \ge 1$ and $(x_1, x_2, \ldots, x_m) \in \mathbf{B}^m$, write

$$g_m(x_1, x_2, \ldots, x_m) = \sum_{n=1}^{m} a_n \left\|\sum_{i=1}^{m} x_i\right\|^q.$$

One can easily check that, for each $m \ge 1$, the function g_m satisfies conditions (5) and (6) with α and β given by (7) . Let

$$T_{m,n} = g_m(V_1, \ldots, V_n, 0, \ldots, 0), \quad Y_{m,n} = g_m(0, \ldots, 0, V_n, 0, \ldots, 0), \quad 1 \le n \le m.$$

Clearly,

$$T_{m,m} = \sum_{n=1}^{m} a_n \left\| \sum_{i=1}^{n} V_i \right\|^q$$

and

$$\max_{1 \le n \le m} Y_{m,n} = \max_{1 \le n \le m} \left(\sum_{i=n}^{m} a_i \right) \|V_n\|^q = \max_{1 \le n \le m} (b_n - b_{m+1}) \|V_n\|^q .$$

Then by Theorem 4 we have for all nonnegative real numbers t,

$$\begin{aligned} \mathbb{P}\left(\max_{1 \le n \le m} (b_n - b_{m+1}) \|V_n\|^q > t \right) &= \mathbb{P}\left(\max_{1 \le n \le m} Y_{m,n} > t \right) \\ &\le 2\mathbb{P}\left(T_{m,m} > \frac{t}{\alpha} \right) \\ &= 2\mathbb{P}\left(\sum_{n=1}^{m} a_n \left\| \sum_{i=1}^{n} V_i \right\|^q > \frac{t}{\alpha} \right), \end{aligned} \tag{12}$$

and by Theorem 6 we have for all nonnegative real numbers s, t, and u,

$$\begin{aligned} \mathbb{P}\left(T_{m,m} > s + t + u \right) \le \mathbb{P}\left(\max_{1 \le n \le m} (b_n - b_{m+1}) \|V_n\|^q > \frac{s}{\beta^2} \right) \\ +4\mathbb{P}\left(T_{m,m} > \frac{u}{\alpha\beta} \right) \mathbb{P}\left(T_{m,m} > \frac{t}{\alpha\beta^2} \right). \end{aligned} \tag{13}$$

Note that with probability 1,

$$T_{m,m} = \sum_{n=1}^{m} a_n \left\| \sum_{i=1}^{n} V_i \right\|^q \nearrow \sum_{n=1}^{\infty} a_n \left\| \sum_{i=1}^{n} V_i \right\|^q$$

and

$$\max_{1 \le n \le m} (b_n - b_{m+1}) \|V_n\|^q \nearrow \sup_{n \ge 1} b_n \|V_n\|^q \quad \text{as} \quad m \to \infty.$$

Thus, letting $m \to \infty$, (8) and (9) follow from (12) and (13) respectively.

We only need to verify (11) since (10) follows from (8). Set

$$\gamma = \alpha + \beta \quad \text{and} \quad T = \sum_{n=1}^{\infty} a_n \left\| \sum_{i=1}^{n} V_i \right\|^q .$$

Let $c > t_0$. Noting $\gamma > 1$, $\gamma/\alpha > 1$, and $\gamma/\beta > 1$, by (9) with $s = t = u = \gamma^3 x$, we have that

$$\begin{aligned}
\mathbb{E}(T) &= 3\gamma^3 \int_0^\infty \mathbb{P}\left(T > 3\gamma^3 x\right) dx \\
&= 3\gamma^3 \left(\int_0^c + \int_c^\infty\right) \mathbb{P}\left(T > 3\gamma^3 x\right) dx \\
&\leq 3\gamma^3 \left(c + \int_c^\infty \mathbb{P}\left(\sup_{n\geq 1} b_n \left\|V_n\right\|^q > x\right) dx + 4\int_c^\infty \mathbb{P}^2(T > x) dx\right) \\
&\leq 3\gamma^3 \left(c + \mathbb{E}\left(\sup_{n\geq 1} b_n \left\|V_n\right\|^q\right) + 4\mathbb{P}(T > c)\int_0^\infty \mathbb{P}(T > x) dx\right) \\
&\leq 3\gamma^3 c + 3\gamma^3 \mathbb{E}\left(\sup_{n\geq 1} b_n \left\|V_n\right\|^q\right) + \frac{1}{2}\mathbb{E}(T)
\end{aligned}$$

since $12\gamma^3 \mathbb{P}(T > c) \leq 1/2$ by the choice of c. We thus conclude that

$$\mathbb{E}(T) \leq 6(\alpha+\beta)^3 \mathbb{E}\left(\sup_{n\geq 1} b_n \left\|V_n\right\|^q\right) + 6(\alpha+\beta)^3 c \ \ \forall\, c > t_0$$

and hence (11) is established. □

3 Proof of Theorem 3

For the proof of Theorem 3, we need the following five preliminary lemmas.

Lemma 1. *Let* $\{c_k;\ k \geq 1\}$ *be a sequence of real numbers such that*

$$\sum_{k=1}^\infty |c_k| < \infty$$

and let $\{a_{n,k};\ k \geq 1, n \geq 1\}$ *be an array of real numbers such that*

$$\sup_{n\geq 1, k\geq 1} |a_{n,k}| < \infty \ \ \textit{and} \ \ \lim_{n\to\infty} a_{n,k} = 0 \ \forall\, k \geq 1.$$

Then

$$\lim_{n\to\infty} \sum_{k=1}^\infty a_{n,k} c_k = 0.$$

Proof. This follows immediately from the Lebesgue dominated convergence theorem with counting measure on the positive integers. $\square$

The proofs of Lemmas 2 and 3 and Theorem 3 involve a symmetrization argument. For the sequence $\{X_n;\ n \geq 1\}$ of independent copies of the **B**-valued random variable X with partial sums $S_n = \sum_{k=1}^{n} X_k$, $n \geq 1$, let $\{X', X_n';\ n \geq 1\}$ be an independent copy of $\{X, X_n;\ n \geq 1\}$. The symmetrized random variables are defined by $\hat{X} = X - X'$, $\hat{X}_n = X_n - X_n'$, $n \geq 1$. Set $S_n' = \sum_{k=1}^{n} X_k'$, $\hat{S}_n = \sum_{k=1}^{n} \hat{X}_k$, $n \geq 1$.

Lemma 2. *Let* $0 < p < 2$ *and let* $\{X_n;\ n \geq 1\}$ *be a sequence of independent copies of a* **B***-valued random variable* X*. Then*

$$\lim_{n\to\infty} \frac{S_n}{n^{1/p}} = 0 \ \ a.s. \tag{14}$$

if and only if

$$\mathbb{E}\|X\|^p < \infty \ \ and \ \ \frac{S_{2^n}}{2^{n/p}} \to_{\mathbb{P}} 0. \tag{15}$$

Proof. By Theorem 1, we see that (15) immediately follows from (14) . We now show that (15) implies (14) . For $0 < p < 1$, (14) follows from (15) since

$$\lim_{n\to\infty} \frac{\sum_{k=1}^{n} \|X_k\|}{n^{1/p}} = 0 \ \text{ a.s. if and only if } \ \mathbb{E}\|X\|^p < \infty.$$

Clearly, for $1 \leq p < 2$, (15) implies that $\mathbb{E}\|X\| < \infty$ and hence by the SLLN of Mourier [15]

$$\frac{S_n}{n} \to \mathbb{E}X \ \text{ a.s.}$$

Then

$$\frac{S_{2^n}}{2^n} \to_{\mathbb{P}} \mathbb{E}X$$

and so $\mathbb{E}X = 0$ in view of the second half of (15) . We thus conclude that when $1 \leq p < 2$, (15) entails $\mathbb{E}X = 0$.

Next, it follows from the second half of (15) that

$$\frac{\hat{S}_{2^n}}{2^{n/p}} \to_{\mathbb{P}} 0.$$

Hence for any given $\epsilon > 0$, there exists a positive integer n_ϵ such that

$$\mathbb{P}\left(\left\|\hat{S}_{2^n}\right\| > 2^{n/p}\epsilon\right) \leq 1/24, \ \ \forall\, n \geq n_\epsilon.$$

Note that $\{\hat{X}_n;\ n \geq 1\}$ is a sequence of i.i.d. **B**-valued random variables. Thus, by the second part of Proposition 6.8 of Ledoux and Talagrand [10, p. 156], we have

$$\mathbb{E}\left\|\hat{S}_{2^n}\right\| \leq 6\mathbb{E}\max_{1\leq i\leq 2^n}\left\|\hat{X}_i\right\| + 6\times 2^{n/p}\epsilon \leq 12\mathbb{E}\max_{1\leq i\leq 2^n}\|X_i\| + 6\times 2^{n/p}\epsilon,\ \ \forall\ n\geq n_\epsilon$$

and hence

$$\frac{\mathbb{E}\left\|\hat{S}_{2^n}\right\|}{2^{n/p}} \leq 12\left(\frac{\mathbb{E}\max_{1\leq i\leq 2^n}\|X_i\|}{2^{n/p}}\right) + 6\epsilon,\ \ \forall\ n\geq n_\epsilon.$$

It is easy to show that, for $1 \leq p < 2$, the first half of (15) implies that

$$\lim_{n\to\infty}\frac{\mathbb{E}\max_{1\leq i\leq 2^n}\|X_i\|}{2^{n/p}} = 0.$$

We thus have that

$$\lim_{n\to\infty}\frac{\mathbb{E}\left\|\hat{S}_{2^n}\right\|}{2^{n/p}} = 0. \tag{16}$$

Since $\mathbb{E}X = 0$, applying (2.5) of Ledoux and Talagrand [10, p. 46], we have that

$$\max_{2^{n-1}\leq m<2^n}\frac{\mathbb{E}\|S_m\|}{m^{1/p}} \leq 2^{1/p}\max_{2^{n-1}\leq m<2^n}\frac{\mathbb{E}\|S_m\|}{2^{n/p}} \leq 2^{1/p}\times\frac{\mathbb{E}\|S_{2^n}\|}{2^{n/p}} \leq 2^{1/p}\times\frac{\mathbb{E}\left\|\hat{S}_{2^n}\right\|}{2^{n/p}}$$

for $n \geq 1$. It now follows from (16) that

$$\lim_{n\to\infty}\frac{\mathbb{E}\|S_n\|}{n^{1/p}} = 0$$

and hence that

$$\frac{S_n}{n^{1/p}} \to_{\mathbb{P}} 0.$$

By Theorem 1 again, we see that (14) follows. □

Lemma 3. *Let* $0 < p < 2$ *and* $0 < q < \infty$. *Let* $\{X_n;\ n \geq 1\}$ *be a sequence of independent copies of a* **B**-*valued random variable* X. *If* (2) *holds, i.e., if*

$$\sum_{n=1}^{\infty}\frac{1}{n}\left(\frac{\|S_n\|}{n^{1/p}}\right)^q < \infty\ \ a.s.,$$

then (14) *holds, i.e.,*

$$\lim_{n\to\infty}\frac{S_n}{n^{1/p}} = 0\ \ a.s.$$

Proof. We first show that (2) implies that

$$\frac{S_{2^n}}{2^{n/p}} \to_{\mathbb{P}} 0. \tag{17}$$

To see this, for $n \geq 1$ and $\mathbf{x} = (x_1, x_2, \ldots, x_{2^n}) \in \mathbf{B}^{2^n}$ write

$$g_n(\mathbf{x}) = g_n(x_1, x_2, \ldots, x_{2^n}) = \sum_{k=2^n}^{2^{n+1}-1} \frac{1}{k} \left(\frac{\left\| \sum_{i=1}^{k+1-2^n} x_i \right\|}{k^{1/p}} \right)^q.$$

Clearly, $g_n : \mathbf{B}^{2^n} \to [0, \infty]$ is a measurable function satisfying condition (6) with β given by (7) . Set

$$V_1 = S_{2^n},\ V_j = X_{2^n+j-1},\ 2 \leq j \leq 2^n,$$

$$M_{n,j} = g_n\left(V_1, \ldots, V_j, 0, \ldots, 0\right),\ D_{n,j} = g_n\left(0, \ldots, 0, -V_j, \ldots, -V_{2^n}\right),\ 1 \leq j \leq 2^n.$$

By Theorem 5 (i.e., Theorem 2.2 of of Li and Rosalsky [13]), we have that

$$\mathbb{P}\left(\max_{1 \leq j \leq 2^n} M_{n,j} > t + u\right) \leq \frac{\mathbb{P}\left(M_{n,2^n} > t/\beta\right)}{1 - \max_{2 \leq j \leq 2^n} \mathbb{P}\left(D_{n,j} > u/\beta\right)},\ \forall\ s \geq 0, u \geq 0. \tag{18}$$

It is easy to see that

$$M_{n,1} = g_n\left(S_{2^n}, 0, \ldots, 0\right) = \left(\sum_{k=2^n}^{2^{n+1}-1} \frac{1}{k^{1+q/p}}\right) \left(\|S_{2^n}\|\right)^q \geq 2^{-1-q/p} \left(\frac{\|S_{2^n}\|}{2^{n/p}}\right)^q \tag{19}$$

and it follows from (2) that

$$M_{n,2^n} = g_n\left(S_{2^n}, X_{2^n+1}, \ldots, X_{2^{n+1}-1}\right) = \sum_{k=2^n}^{2^{n+1}-1} \frac{1}{k} \left(\frac{\|S_k\|}{k^{1/p}}\right)^q \to 0\ \text{ a.s.} \tag{20}$$

Since $\{X_n;\ n \geq 1\}$ is a sequence of independent copies of X, we have that for all $u \geq 0$,

$$\mathbb{P}\left(D_{n,j} > u\right) = \mathbb{P}\left(g_n\left(0, \ldots, 0, X_1, \ldots, X_{2^n-j+1}\right) > u\right),\ \ 2 \leq j \leq 2^n.$$

Note that

$$g_n\left(0, \ldots, 0, X_1, X_2, \ldots, X_{2^n-j+1}\right) \leq g_n\left(0, \ldots, 0, X_1, X_2, \ldots, X_{2^n-j+2}\right),\ \ 2 \leq j \leq 2^n.$$

We thus conclude that for all $u \geq 0$,

$$\max_{2 \leq j \leq 2^n} \mathbb{P}\left(D_{n,j} > u/\beta\right) \leq \mathbb{P}\left(g_n\left(X_1, X_2, \ldots, X_{2^n}\right) > u/\beta\right). \tag{21}$$

Set

$$a_{n,k} = \begin{cases} \left(\frac{k}{2^n}\right)^{1+q/p} & \text{if } 1 \leq k \leq 2^n \\ 0 & \text{if } k > 2^n. \end{cases}$$

Then clearly $\{a_{n,k};\ k \geq 1, n \geq 1\}$ is an array of nonnegative real numbers such that

$$\sup_{n \geq 1, k \geq 1} a_{n,k} \leq 1 < \infty \text{ and } \lim_{n \to \infty} a_{n,k} = 0 \ \ \forall\, k \geq 1.$$

Note that, for $n \geq 1$,

$$\begin{aligned} g_n(X_1, X_2, \ldots, X_{2^n}) &= \sum_{k=2^n}^{2^{n+1}-1} \frac{1}{k} \left(\frac{\left\| \sum_{i=1}^{k+1-2^n} X_i \right\|}{k^{1/p}} \right)^q \\ &\leq \sum_{j=1}^{2^n} \frac{1}{2^n} \left(\frac{\|S_j\|}{2^{n/p}} \right)^q \\ &= \sum_{j=1}^{2^n} \left(\frac{j}{2^n} \right)^{1+q/p} \left(\frac{1}{j} \left(\frac{\|S_j\|}{j^{1/p}} \right)^q \right) \\ &= \sum_{k=1}^{\infty} a_{n,k} \left(\frac{1}{k} \left(\frac{\|S_k\|}{k^{1/p}} \right)^q \right). \end{aligned}$$

Then, by Lemma 1, (2) implies that

$$\lim_{n \to \infty} g_n(X_1, X_2, \ldots, X_{2^n}) = 0 \text{ a.s.} \tag{22}$$

It now follows from (18) and (20)–(22) that

$$\lim_{n \to \infty} \mathbb{P}(M_{n,1} > \epsilon) \leq \lim_{n \to \infty} \frac{\mathbb{P}\left(M_{n,2^n} > \frac{\epsilon}{2\beta}\right)}{1 - \mathbb{P}\left(g_n(X_1, X_2, \ldots, X_{2^n}) > \frac{\epsilon}{2\beta}\right)} = 0 \ \ \forall\, \epsilon > 0;$$

that is,

$$M_{n,1} \to_{\mathbb{P}} 0$$

and hence (17) follows from (19).

We now show that (2) implies that

$$\mathbb{E}\|X\|^p < \infty. \tag{23}$$

To see this, (2) clearly ensures that

$$\sum_{n=1}^{\infty} a_n \left\|\hat{S}_n\right\|^q = \sum_{n=1}^{\infty} \frac{1}{n} \left(\frac{\|\hat{S}_n\|}{n^{1/p}}\right)^q \leq \beta \left(\sum_{n=1}^{\infty} \frac{1}{n} \left(\frac{\|S_n\|}{n^{1/p}}\right)^q + \sum_{n=1}^{\infty} \frac{1}{n} \left(\frac{\|S_n'\|}{n^{1/p}}\right)^q\right) < \infty \text{ a.s.,} \tag{24}$$

where $a_n = n^{-1-q/p},\ n \geq 1$. Since $\{\hat{X}_n;\ n \geq 1\}$ is a sequence of independent copies of the **B**-valued random variable $\hat{X}$, it follows from (8) of Theorem 7 that

$$\mathbb{P}\left(\sup_{n\geq 1} b_n \left\|\hat{X}_n\right\|^q > t\right) \leq 2\mathbb{P}\left(\sum_{n=1}^{\infty} \frac{1}{n} \left(\frac{\|\hat{S}_n\|}{n^{1/p}}\right)^q > \frac{t}{\alpha}\right) \quad \forall\ t \geq 0,$$

where

$$b_n = \sum_{k=n}^{\infty} n^{-1-q/p},\ n \geq 1$$

which, together with (24), ensures that

$$\sup_{n\geq 1} b_n \left\|\hat{X}_n\right\|^q < \infty \text{ a.s.} \tag{25}$$

It is easy to check that

$$\lim_{n\to\infty} \frac{b_n}{n^{-q/p}} = \frac{p}{q},$$

and so we have by (25) that

$$\left(\sup_{n\geq 1} \frac{\left\|\hat{X}_n\right\|}{n^{1/p}}\right)^q = \sup_{n\geq 1} n^{-q/p} \left\|\hat{X}_n\right\|^q < \infty \text{ a.s.}$$

Since the $\hat{X}_n, n \geq 1$ are i.i.d., it follows from the Borel-Cantelli lemma that for some finite $\lambda > 0$,

$$\sum_{n=1}^{\infty} \mathbb{P}\left(\|\hat{X}\| > \lambda n^{1/p}\right) < \infty$$

and hence

$$\mathbb{E}\|X - X'\|^p < \infty$$

which is equivalent to (23) . By Lemma 2, (14) now follows from (17) and (23) . The proof of Lemma 3 is complete. □

Lemma 4. *Let* $(\mathbf{E}, \mathscr{G})$ *be a measurable linear space and* $g : \mathbf{E} \to [0, \infty]$ *be a measurable even function such that for all* $\mathbf{x}, \mathbf{y} \in \mathbf{E}$*,*

$$g(\mathbf{x} + \mathbf{y}) \le \beta\,(g(\mathbf{x}) + g(\mathbf{y}))\,,$$

where $1 \le \beta < \infty$ *is a constant, depending only on the function* g*. If* $\mathbf{V}$ *is an* $\mathbf{E}$*-valued random variable and* $\hat{\mathbf{V}}$ *is a symmetrized version of* $\mathbf{V}$ *(i.e.,* $\hat{\mathbf{V}} = \mathbf{V} - \mathbf{V}'$ *where* $\mathbf{V}'$ *is an independent copy of* $\mathbf{V}$*), then for all* $t \ge 0$*, we have that*

$$\mathbb{P}(g(\mathbf{V}) \le t)\mathbb{E}g(\mathbf{V}) \le \beta\mathbb{E}g(\hat{\mathbf{V}}) + \beta t \tag{26}$$

and

$$\mathbb{E}g(\hat{\mathbf{V}}) \le 2\beta\mathbb{E}g(\mathbf{V}). \tag{27}$$

Moreover, if

$$g(\mathbf{V}) < \infty \quad a.s., \tag{28}$$

then

$$\mathbb{E}g(\mathbf{V}) < \infty \quad \textit{if and only if} \quad \mathbb{E}g(\hat{\mathbf{V}}) < \infty. \tag{29}$$

Proof. We only give the proof of the second part of this lemma since the first part of this lemma is a special case of Lemma 3.2 of Li and Rosalsky [13]. Note that, by (28), there exists a finite positive number τ such that

$$\mathbb{P}(g(\mathbf{V}) \le \tau) \ge 1/2.$$

It thus follows from (26) and (27) that

$$\frac{1}{2\beta}\mathbb{E}g(\hat{\mathbf{V}}) \le \mathbb{E}g(\mathbf{V}) \le 2\beta\mathbb{E}g(\hat{\mathbf{V}}) + 2\beta\tau$$

which ensures that (29) holds. □

The following nice result is Proposition 3 of Hechner and Heinkel [7].

Lemma 5 (Hechner and Heinkel [7]). *Let $p > 1$ and let $\{X_n;\ n \geq 1\}$ be a sequence of independent copies of a* **B**-*valued random variable X. Write*

$$u_n = \inf\left\{t:\ \mathbb{P}(\|X\| > t) < \frac{1}{n}\right\},\ n \geq 1.$$

Then the following three statements are equivalent:

(i) $\displaystyle\int_0^\infty \mathbb{P}^{1/p}(\|X\| > t)dt < \infty;$

(ii) $\displaystyle\sum_{n=1}^{\infty} \frac{u_n}{n^{1+1/p}} < \infty;$

(iii) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^{1+1/p}} \mathbb{E}\left(\max_{1\leq k\leq n} \|X_k\|\right) < \infty.$

Proof of Theorem 3. Firstly, we see that (1) immediately implies that (2) holds. Thus, by Lemma 3, for $0 < q < \infty$, each of (1) and (2) implies that (4) holds.

Secondly, we show that (1) follows from (2) and (3). To see this, by Lemma 4, we conclude that (1) is equivalent to

$$\sum_{n=1}^{\infty} \frac{1}{n} \mathbb{E}\left(\frac{\|\hat{S}_n\|}{n^{1/p}}\right)^q < \infty. \tag{30}$$

Since (2) ensures that (24) holds, by (10) and (11) of Theorem 7, we see that (30) holds if and only if

$$\mathbb{E}\left(\sup_{n\geq 1} b_n \left\|\hat{X}_n\right\|^q\right) < \infty, \tag{31}$$

where $b_n = \sum_{k=n}^{\infty} n^{-1-q/p}, n \geq 1$. Since $\lim_{n\to\infty} b_n/n^{-q/p} = p/q$, we conclude that (31) is equivalent to

$$\mathbb{E}\left(\sup_{n\geq 1} \frac{\left\|\hat{X}_n\right\|^p}{n}\right)^{q/p} = \mathbb{E}\left(\sup_{n\geq 1} \frac{\left\|\hat{X}_n\right\|^q}{n^{q/p}}\right) < \infty. \tag{32}$$

Note that we have from (3) that

$$\begin{cases} \mathbb{E}\|X\|^p < \infty & \text{if } 0 < q < p, \\ \mathbb{E}\|X\|^p \ln(1 + \|X\|) < \infty & \text{if } q = p, \\ \mathbb{E}\|X\|^q < \infty & \text{if } q > p \end{cases}$$

which is equivalent to

$$\begin{cases} \mathbb{E}\|\hat{X}\|^p < \infty & \text{if } 0 < q < p, \\ \mathbb{E}\|\hat{X}\|^p \log(1 + \|\hat{X}\|) < \infty & \text{if } q = p, \\ \mathbb{E}\|\hat{X}\|^q < \infty & \text{if } q > p. \end{cases} \tag{33}$$

Burkholder [2] proved that (33) and (32) are equivalent. We thus conclude that (1) follows from (2) and (3) .

Since (1) and (30) are equivalent, (30) implies that (32) holds, and (32) and (33) are equivalent, we conclude that (3) follows from (1) if $q \geq p$.

We now show that (1) implies that (3) holds if $0 < q < p$. By the Lévy inequality, we have that, for every $n \geq 1$ and all $t \geq 0$,

$$\mathbb{P}\left(\max_{1\leq k\leq n} \left\|\hat{X}_k\right\|^q > t\right) = \mathbb{P}\left(\max_{1\leq k\leq n} \left\|\hat{X}_k\right\| > t^{1/q}\right)$$
$$\leq 2\mathbb{P}\left(\left\|\hat{S}_n\right\| > t^{1/q}\right) = 2\mathbb{P}\left(\left\|\hat{S}_n\right\|^q > t\right),$$

which ensures that, for every $n \geq 1$,

$$\mathbb{E}\left(\max_{1\leq k\leq n} \left\|\hat{X}_k\right\|^q\right) \leq 2\mathbb{E}\left\|\hat{S}_n\right\|^q. \tag{34}$$

Since (1) and (30) are equivalent, it now follows from (1) and (34) that

$$\sum_{n=1}^{\infty} \frac{1}{n^{1+1/p_1}} \mathbb{E}\left(\max_{1\leq k\leq n} Y_k\right) = \sum_{n=1}^{\infty} \frac{1}{n^{1+q/p}} \mathbb{E}\left(\max_{1\leq k\leq n} \left\|\hat{X}_k\right\|^q\right) < \infty, \tag{35}$$

where $p_1 = p/q > 1$ (since $0 < q < p$) and $Y = \|\hat{X}\|^q,\ Y_n = \left\|\hat{X}_n\right\|^q,\ n \geq 1$. By Lemma 5, (35) is equivalent to

$$\int_0^{\infty} \mathbb{P}^{1/p_1}(Y > t)dt < \infty,$$

i.e.,

$$\int_0^{\infty} \mathbb{P}^{q/p}\left(\|X - X'\|^q > t\right) dt < \infty. \tag{36}$$

Let $m(\|X\|)$ denote a median of $\|X\|$. Since, by the weak symmetrization inequality, we have that

$$\mathbb{P}(|\|X\| - m(\|X\|)| > t) \leq 2\mathbb{P}\left(\left|\|X\| - \|X'\|\right| > t\right)$$
$$\leq 2\mathbb{P}\left(\|X - X'\| > t\right) \leq 4\mathbb{P}\left(\|X\| > \frac{t}{2}\right) \quad \forall\, t \geq 0,$$

we conclude that (36) is equivalent to

$$\int_0^\infty \mathbb{P}^{q/p}\left(\|X\|^q > t\right) dt < \infty,$$

i.e., (3) holds if $0 < q < p$.

Finally, by Lemma 3, (2) implies that $\mathbb{E}\|X\|^p < \infty$. Then (32) holds and hence (30) holds if $0 < q < p$. Since, under (2) , (1) and (30) are equivalent, we see that (1) follows from (2) if $0 < q < p$. □

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