Spectral Radii of Truncated Circular Unitary Matrices

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Abstract

Consider a truncated circular unitary matrix which is a p_n by p_n submatrix of an n by n circular unitary matrix by deleting the last $n - p_n$ columns and rows. Jiang and Qi (2017) proved that the maximum absolute value of the eigenvalues (known as spectral radius) of the truncated matrix, after properly normalized, converges in distribution to the Gumbel distribution if p_n/n is bounded away from 0 and 1. In this paper we investigate the limiting distribution of the spectral radius under one of the following four conditions: (1). $p_n \to \infty$ and $p_n/n \to 0$ as $n \to \infty$; (2). $(n - p_n)/n \to 0$ and $(n - p_n)/(\log n)^3 \to \infty$ as $n \to \infty$; (3). $n - p_n \to \infty$ and $(n - p_n)/\log n \to 0$ as $n \to \infty$ and (4). $n - p_n = k \geq 1$ is a fixed integer. We prove that the spectral radius converges in distribution to the Gumbel distribution under the first three conditions and to a reversed Weibull distribution under the fourth condition.

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1 Introduction

The early study of large random matrices was stimulated by analysis of high-dimensional data. One example is Wishart's (1928) investigation on large covariance matrices whose statistical properties are mainly determined by eigenvalues and eigenvectors from the point view of a principal components analysis. Since then, the random matrix theory has been developed very rapidly and found many applications in areas such as heavy-nuclei atoms (Wigner, 1955), number theory (Mezzadri and Snaith, 2005), quantum mechanics (Mehta, 2004), condensed matter physics (Forrester, 2010), wireless communications (Couillet and Debbah, 2011).

The study of random matrices has greatly been motivated by Tracy and Widom's (1994, 1996) work. They show that the largest eigenvalues of the three Hermitian matrices (Gaussian orthogonal ensemble, Gaussian unitary ensemble and Gaussian symplectic ensemble) converge to some special distributions that are now known as the Tracy-Widom laws. Subsequently, the Tracy-Widom laws have found their applications in the study of problems such as the longest increasing subsequence (Baik et al., 1999), combinatorics, growth processes, random tilings and the determinantal point processes (see, e.g., Tracy and Widom (2002), Johansson (2007) and references therein) and the largest eigenvalues in the high-dimensional statistics (see, e.g., Johnstone (2001, 2008) and Jiang (2009)). Some recent research focuses on the universality of the largest eigenvalues of matrices with non-Gaussian entries; see, for example, Tao and Vu (2011), Erdős et al. (2012) and the references therein.

Consider a non-Hermitian matrix $\mathbf{M}$ with eigenvalues $z_1, \cdots, z_n$. The largest absolute values of the eigenvalues $\max_{1\le j\le n} |z_j|$ is refereed to as the spectral radius of $\mathbf{M}$. The spectral radii of the real, complex and symplectic Ginibre ensembles are investigated by Rider (2003, 2004) and Rider and Sinclair (2014), and it is proved that the spectral radius for the complex Ginibre ensemble converges to the Gumbel distribution. This indicates that non-Hermitian matrices exhibit quite different behaviors from Hermitian matrices in terms of the limiting distribution for the largest absolute values of the eigenvalues.

A very recent paper by Jiang and Qi (2017) studies the largest radii of three rotation-invariant and non-Hermitian random matrices: the spherical ensemble, the truncation of circular unitary ensemble and the product of independent complex Ginibre ensembles. It is proved in the paper that the spectral radii converge to the Gumbel distribution and some new distributions.

The circular unitary ensemble is an $n \times n$ random matrix with Haar measure on the unitary group, and it is also called Haar-invariant unitary matrix. Let $\mathbf{U}$ be an $n \times n$ circular unitary matrix. The n eigenvalues of the circular unitary matrix $\mathbf{U}$ are distributed over $\{z \in \mathcal{C} : |z| = 1\}$, where $\mathcal{C}$ is the complex plane, and their joint density function is given by

$$\frac{1}{n!(2\pi)^n} \cdot \prod_{1\le j<k\le n} |z_j - z_k|^2;$$

see, e.g., Hiai and Petz (2000).

For $n > p \geq 1$, write

$$\mathbf{U} = \begin{pmatrix} \mathbf{A} & \mathbf{C}^* \\ \mathbf{B} & \mathbf{D} \end{pmatrix}$$

where $\mathbf{A}$, as a truncation of $\mathbf{U}$, is a $p \times p$ submatrix. Let $z_1, \cdots, z_p$ be the eigenvalues of $\mathbf{A}$. Then their density function is

$$C \cdot \prod_{1 \leq j < k \leq p} |z_j - z_k|^2 \prod_{j=1}^{p} (1 - |z_j|^2)^{n-p-1} \tag{1.1}$$

where C is a normalizing constant. See, e.g., Zyczkowski and Sommers (2000).

Assume $p = p_n$ depends on n and set $c = \lim_{n\to\infty} \frac{p_n}{n}$. Życzkowski and Sommers (2000) show that the empirical distribution of z_i's converges to the distribution with density proportional to $\frac{1}{(1-|z|^2)^2}$ for $|z| \leq c$ if $c \in (0, 1)$. Dong et al. (2012) prove that the empirical distribution goes to the circular law and the arc law as $c = 0$ and $c = 1$, respectively. See also Diaconis and Evans (2001) and Jiang (2009, 2010) and references therein for more results.

Jiang and Qi (2017) have proved that the spectral radius $\max_{1\leq j\leq p} |z_j|$ for the truncated circular unitary ensemble converges to the Gumbel distribution when the dimension of the truncated truncated circular unitary matrix is of the same order as the dimension of the original circular unitary matrix, see Theorem 1 in section 2.

In this paper we consider heavily truncated and lightly truncated circular unitary matrices and investigate the limiting distribution of the spectral radii for those truncated circular unitary matrices. Our results complement that in Jiang and Qi (2017).

The rest of the paper is organized as follows. The main results in this paper are given in section 2 and their proofs are provided in section 3.

2 Main Results

Consider the $p_n \times p_n$ submatrix $\mathbf{A}$, truncated from a $n \times n$ circular unitary matrix $\mathbf{U}$ in section 1. Denote the p_n eigenvalues as $z_1, \cdots, z_{p_n}$ with the joint density function given by (1.1).

For completeness, we first quote a theorem in Jiang and Qi (2017) on the limiting distribution of the spectral radii $\max_{1\leq j\leq p_n} |z_j|$ before we give our results in the paper.

Theorem 1 *Assume that $z_1, \cdots, z_{p_n}$ have density as in (1.1) and there exist constants $h_1, h_2 \in (0, 1)$ such that $h_1 < \frac{p_n}{n} < h_2$ for all $n \geq 2$. Then $(\max_{1\leq j\leq p_n} |z_j| - A_n)/B_n$ converges weakly to the Gumbel distribution $\Lambda(x) = \exp(-e^{-x})$, $x \in \mathbb{R}$, where $A_n = c_n +$*

$\frac{1}{2}(1-c_n^2)^{1/2}(n-1)^{-1/2}a_n$, $B_n = \frac{1}{2}(1-c_n^2)^{1/2}(n-1)^{-1/2}b_n$,

$$c_n = \Big(\frac{p_n - 1}{n-1}\Big)^{1/2}, \quad b_n = b\Big(\frac{nc_n^2}{1-c_n^2}\Big), \quad a_n = a\Big(\frac{nc_n^2}{1-c_n^2}\Big)$$

with

$$a(y) = (\log y)^{1/2} - (\log y)^{-1/2}\log(\sqrt{2\pi}\log y) \quad \textit{and} \quad b(y) = (\log y)^{-1/2}$$

for $y > 3$.

Note that in Theorem 1, a restriction on the dimension p_n of the truncated circular unitary matrix $\mathbf{A}$ is made as follows: there exist some $0 < h_1 < h_2 < 1$ such that $h_1 n < p_n < h_2 n$ for all large n. In this paper we are devoted to study of the spectral radii $\max_{1\le j\le p_n}|z_j|$ in the following conditions:

$$p_n \to \infty \text{ and } \frac{p_n}{n} \to 0 \text{ as } n \to \infty; \tag{2.1}$$

$$\frac{n-p_n}{(\log n)^3} \to \infty \text{ and } \frac{n-p_n}{n} \to 0 \text{ as } n \to \infty; \tag{2.2}$$

$$n - p_n \to \infty \text{ and } \frac{n-p_n}{\log n} \to 0 \text{ as } n \to \infty; \tag{2.3}$$

$$n - p_n = k \ge 1 \text{ is fixed integer.} \tag{2.4}$$

The main results of the paper are the following theorems:

Theorem 2 *Under condition* (2.1) *or* (2.2), $(\max_{1\le j\le p_n}|z_j| - A_n)/B_n$ *converges weakly to the Gumbel distribution* $\Lambda(x) = \exp(-e^{-x})$, $x \in \mathbb{R}$, *where* A_n *and* B_n *are defined as in Theorem 1.*

Theorem 3 *Under condition* (2.3), $(\max_{1\le j\le p_n}|z_j| - A_n)/B_n$ *converges weakly to the Gumbel distribution* $\Lambda(x) = \exp(-e^{-x})$, $x \in \mathbb{R}$, *where* $A_n = (1 - a_n/n)^{1/2}$ *and* $B_n = a_n/(2nk_n)$, *where* a_n *is given by*

$$\frac{1}{(k_n-1)!}\int_0^{a_n} t^{k_n-1}e^{-t}dt = \frac{k_n}{n}.$$

where $k_n = n - p_n$.

Theorem 4 *Under condition* (2.4), $\frac{2n^{1+1/k}}{((k+1)!)^{1/k}}(\max_{1\le j\le p_n}|z_j| - 1)$ *converges weakly to the reversed Weibull distribution* $W_k(x)$ *defined as*

$$W_k(x) = \begin{cases} \exp(-(-x)^k), & x \le 0; \\ 1, & x > 0. \end{cases}$$

We notice that the limiting distribution of the spectral radii depends on the dimension of truncated matrices. Our results in Theorems 2, 3 and 4 indicate that the limiting distribution of the spectral radii of the truncated circular unitary matrices is Gumbel distribution Λ if the parameter $k_n = n - p_n$, the number of truncated columns and rows diverges. When the truncation is very light, that is, $k_n = n - p_n = k \geq 1$ is a fixed integer, the limiting distribution of the spectral radii of the truncated matrices is the reversed Weibull distribution W_k.

It is obvious that the case when $k_n = n - p_n$ is of order between $\log n$ and $(\log n)^3$ has not been covered in Theorems 1 to 4. We conjecture that $\max_{1\leq j\leq p_n} |z_j|$, after properly normalized, converges in distribution to the Gumbel distribution in this case.

3 Proofs

Set $k_n = n - p_n$ and define $a(x)$ and $b(x)$ as in Theorem 1. Define $\phi(x) = \frac{1}{\sqrt{2\pi}}e^{-x^2/2}$ and $\Phi(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{x} e^{-t^2/2}\,dt$ for $x \in \mathbb{R}$, the density function and the cumulative distribution of the standard normal, respectively. The symbol $C_n \sim D_n$ as $n \to \infty$ means that $\lim_{n\to\infty} \frac{C_n}{D_n} = 1$.

For random variables $\{X_n;\, n \geq 1\}$ and constants $\{a_n;\, n \geq 1\}$, we write $X_n = O_p(a_n)$ if $\lim_{x\to+\infty} \limsup_{n\to\infty} P(|\frac{X_n}{a_n}| \geq x) = 0$. It is well known that $\frac{X_n}{a_n b_n} \to 0$ in probability as $n \to \infty$ if $X_n = O_p(a_n)$ and $\{b_n;\, n \geq 1\}$ is a sequence of constants with $\lim_{n\to\infty} b_n = \infty$.

Let U_i, $i \geq 1$ be a sequence of i.i.d. random variables uniformly distributed over $(0, 1)$, and $U_{1:n} \leq U_{2:n} \leq \cdots \leq U_{n:n}$ be the order statistics of $U_1, U_2, \cdots, U_n$ for each $n \geq 1$. Then from page 14 on the book by Balakrishnan and Cohen (1991), we know that the cumulative distribution function of $U_{i:n}$ is given by

$$F_{i:n}(x) = \sum_{r=i}^{n} \binom{n}{r} x^r (1-x)^{n-r} = \frac{n!}{(i-1)!(n-i)!}\int_0^x t^{i-1}(1-t)^{n-i}dt,\ 0 \leq x \leq 1 \tag{3.1}$$

for each $1 \leq i \leq n$, and the probability density function (pdf) of $U_{i:n}$ is given by

$$f_{i:n}(x) = \frac{n!}{(i-1)!(n-i)!}x^{i-1}(1-x)^{n-i}, \quad 0 < x < 1. \tag{3.2}$$

This is the so-called Beta distribution, denoted by Beta$(i, n-i+1)$.

From (3.2), $U_{p_n-j+1:n-j}$ has a Beta$(p_n - j + 1, k_n)$ distribution with pdf given by

$$f_{p_n-j+1:n-j}(x) = \frac{(n-j)!}{(p_n-j)!(k_n-1)!}x^{p_n-j}(1-x)^{k_n-1}, \qquad x \in (0, 1).$$

For each $n \geq 2$, let $\{Y_{nj};\, 1 \leq j \leq p_n\}$ be independent random variables such that Y_{nj} and $(U_{p_n-j+1:n-j})^{1/2}$ have the same distribution. Jiang and Qi (2017) have shown that

$\max_{1\le j\le p_n}|z_j|$ and $\max_{1\le j\le p_n} Y_{nj}$ have the same distribution, that is

$$P\Big(\max_{1\le j\le p_n}|z_j|^2\le t\Big)=P\Big(\max_{1\le j\le p_n}Y_{nj}^2\le t\Big)=\prod_{j=1}^{p_n}F_{p_n-j+1:n-j}(t). \tag{3.3}$$

for any $0<t<1$, and

$$1-F_{1:k_n}(x)\le 1-F_{2:k_n+1}(x)\le\cdots\le 1-F_{p_n:n-1}(x) \tag{3.4}$$

for $x\in(0,1)$. See the proof of Theorem 2 in Jiang and Qi (2017).

3.1 Preliminary Lemmas

We will present some useful lemmas before we prove our main results.

Lemma 3.1 *Suppose $\{l_n;\, n\ge 1\}$ is sequence of positive integers. Let $z_{nj}\in[0,1)$ be real numbers for $1\le j\le l_n$ such that $\max_{1\le j\le l_n} z_{nj}\to 0$ as $n\to\infty$. Then $\lim_{n\to\infty}\prod_{j=1}^{l_n}(1-z_{nj})\in(0,1)$ exists if and only if the limit $\lim_{n\to\infty}\sum_{j=1}^{l_n} z_{nj}=: z\in(0,\infty)$ exists and the relationship of the two limits is given by*

$$\lim_{n\to\infty}\prod_{i=1}^{l_n}(1-z_{ni})=e^{-z}. \tag{3.5}$$

Proof. From the Taylor expansion

$$\log(1-x)=-x+O(x^2)\qquad \text{as } x\to 0,$$

which implies $\log(1-z_{nj})=-z_{nj}+O(z_{nj}^2)$ uniformly over $1\le j\le l_n$ since $\max_{1\le j\le l_n} z_{nj}\to 0$ as $n\to\infty$. Therefore,

$$\prod_{j=1}^{l_n}(1-z_{nj})=\exp\Big(\sum_{j=1}^{l_n}\log(1-z_{nj})\Big)=\exp\Big(-(1+O(\max_{1\le j\le l_n} z_{nj}))\sum_{j=1}^{l_n} z_{nj}\Big)$$

The lemma can be easily concluded from the above expression. ■

Lemma 3.2 *Let $\{l_n\}$ be a sequence of positive integers such that $l_n\to\infty$ and for each n, $\{z_{nj},\ 1\le j\le l_n\}$ are non-negative numbers such that z_{nj} is non-increasing in j with $z_{n1}>0$. Then for any sequence of positive integers $\{r_n\}$satisfying that $r_n<l_n$ for all large n and $r_n/l_n\to 1$ as $n\to\infty$, we have*

$$\frac{\sum_{j=1}^{l_n} z_{nj}}{\sum_{j=1}^{r_n} z_{nj}}\to 1$$

as $n\to\infty$.

Proof. It suffices to show that

$$\frac{\sum_{j=r_n+1}^{l_n} z_{nj}}{\sum_{j=1}^{r_n} z_{nj}} \to 0 \tag{3.6}$$

as $n \to \infty$. In fact, from the monotonicity of z_{nj}, we have $z_{nj} \le \frac{1}{r_n}\sum_{j=1}^{r_n} z_{nj}$ for $r_n + 1 \le j \le l_n$. Hence

$$\sum_{j=r_n+1}^{l_n} z_{nj} \le \frac{l_n - r_n}{r_n}\sum_{j=1}^{r_n} z_{nj}$$

which implies

$$\frac{\sum_{j=r_n+1}^{l_n} z_{nj}}{\sum_{j=1}^{r_n} z_{nj}} \le \frac{l_n - r_n}{r_n} \to 0,$$

proving (3.6). ■

For the rest of the proofs, define

$$z_{nj} = 1 - F_{p_n-j+1:n-j}(t_n), \qquad 1 \le j \le p_n, \tag{3.7}$$

where $t_n \in (0,1)$ will be specified later in the proof of each theorem. From (3.4),

$$z_{n1} \ge z_{n2} \ge \cdots \ge z_{np_n} \ge 0. \tag{3.8}$$

Obviously, we have for $1 \le j \le p_n$.

$$\begin{aligned} z_{nj} &= \frac{(n-j)!}{(p_n-j)!(k_n-1)!}\int_{t_n}^{1} t^{p_n-j}(1-t)^{k_n-1}dt \\ &= \frac{(n-j)!}{(p_n-j)!(k_n-1)!}\int_{0}^{1-t_n} (1-t)^{p_n-j}t^{k_n-1}dt. \end{aligned} \tag{3.9}$$

Lemma 3.3 *Assume that $1 \le p_n < n$ and $p_n \to \infty$ as $n \to \infty$. Let $\{r_n\}$ satisfy the condition in Lemma 3.2 with $l_n = p_n$. Assume $\alpha_n > 0$ and β_n are real numbers such that $\lim_{n\to\infty} P(Y_{n1}^2 > \beta_n + \alpha_n x) = 0$ for any $x \in R$. If $(\max_{1\le j\le r_n} Y_{nj}^2 - \beta_n)/\alpha_n$ converges in distribution to a cdf G, then $(\max_{1\le j\le p_n} Y_{nj}^2 - \beta_n)/\alpha_n$ converges in distribution to the same distribution G.*

Proof. Note that $(\max_{1\le j\le r_n} Y_{nj}^2 - \beta_n)/\alpha_n$ converges in distribution to the cdf G if and only if

$$\lim_{n\to\infty} P(\max_{1\le j\le r_n} Y_{nj}^2 \le \beta_n + \alpha_n x) = G(x) \tag{3.10}$$

for every continuity point x of G with $G(x) \in (0,1)$. We need to prove the above expression is still true when r_n is replaced by p_n. Now fix x, a continuity point of G with $G(x) \in (0,1)$. Set $t_n = t_n(x) = \beta_n + \alpha_n x$ and define z_{nj} as in (3.7). Note that (3.8) holds, $z_{n1} \to 0$ as $n \to \infty$,

$$P(\max_{1\le j\le r_n} Y_{nj}^2 \le \beta_n + \alpha_n x) = \prod_{j=1}^{r_n}(1 - z_{nj})$$

and

$$P(\max_{1\le j\le p_n} Y_{nj}^2 \le \beta_n + \alpha_n x) = \prod_{j=1}^{p_n}(1 - z_{nj}).$$

By using Lemma 3.1 and (3.10) we have $\sum_{j=1}^{r_n} z_{nj} \to z = -\log G(x)$ which, together with Lemma 3.2, implies $\sum_{j=1}^{p_n} z_{nj} \to z = -\log G(x)$. Once again we have from Lemma 3.1 that

$$\lim_{n\to\infty} P(\max_{1\le j\le p_n} Y_{nj}^2 \le \beta_n + \alpha_n x) = \prod_{j=1}^{p_n}(1 - z_{nj}) = e^{-z} = G(x).$$

This completes the proof of the lemma. ■

Lemma 3.4 *Let Z_n be nonnegative random variables such that $(Z_n^2 - \beta_n)/\alpha_n$ converges weakly to a cdf $G(x)$, where $\alpha_n > 0$ and $\beta_n > 0$ are constants satisfying that $\lim_{n\to\infty} \alpha_n/\beta_n = 0$. Then*

$$\frac{Z_n - \beta_n^{1/2}}{\alpha_n/(2\beta_n^{1/2})} \text{ converges weakly to } G. \tag{3.11}$$

Proof. Set $W_n = (Z_n^2 - \beta_n)/\alpha_n$. We have $Z_n^2 = \beta_n + \alpha_n W_n = \beta_n(1 + \frac{\alpha_n}{\beta_n} W_n)$. Then by Taylor's expansion

$$Z_n = \beta_n^{1/2}(1 + \frac{\alpha_n}{\beta_n} W_n)^{1/2} = \beta_n^{1/2}(1 + \frac{\alpha_n}{2\beta_n} W_n + O_p(\frac{\alpha_n}{\beta_n})^2)$$

and thus we have

$$\frac{Z_n - \beta_n^{1/2}}{\alpha_n/(2\beta_n^{1/2})} = W_n + O_p(\frac{\alpha_n}{\beta_n}),$$

which implies (3.11). ■

Lemma 3.5 *(Lemma 2.2 of Jiang and Qi (2017)) Let $\{j_n, n \ge 1\}$ and $\{x_n, n \ge 1\}$ be positive numbers with $\lim_{n\to\infty} x_n = \infty$ and $\lim_{n\to\infty} j_n x_n^{-1/2}(\log x_n)^{1/2} = \infty$. For fixed $y \in \mathbb{R}$, if $\{c_{n,j}, 1 \le j \le j_n, n \ge 1\}$ are real numbers such that $\lim_{n\to\infty} \max_{1\le j\le j_n} |c_{n,j} x_n^{1/2} - 1| = 0$, then*

$$\lim_{n\to\infty} \sum_{j=1}^{j_n} \big(1 - \Phi((j-1)c_{n,j} + a(x_n) + b(x_n)y)\big) = e^{-y}; \tag{3.12}$$

$$\lim_{n\to\infty} \sum_{j=1}^{j_n} \frac{1}{(j-1)c_{n,j} + a(x_n) + b(x_n)y} \phi((j-1)c_{n,j} + a(x_n) + b(x_n)y\big) = e^{-y}. \tag{3.13}$$

Lemma 3.6 *(Lemma 2.3 of Jiang and Qi (2017) or Proposition 2.10 of Reiss (1981)) Let $\mathcal{B}$ be the collection of all Borel sets on $\mathbb{R}$. Then there exists a constant $C > 0$ such that* *for all $r > k \geq 1$,*

$$\begin{aligned} & \sup_{B\in\mathcal{B}} \Big| P\Big(\frac{r^{3/2}}{\sqrt{(r-k)k}}\big(U_{r-k+1:r} - \frac{r-k}{r}\big) \in B\Big) - \int_B (1 + l_1(t) + l_2(t))\phi(t)dt \Big| \\ \leq \;& C \cdot \Big(\frac{r}{(r-k)k}\Big)^{3/2} \end{aligned}$$

where for $i = 1, 2$, $l_i(t)$ is a polynomial in t of degree $\leq 3i$, depending on r and k, and all of its coefficients are of order $O(\big(\frac{r}{(r-k)k}\big)^{i/2})$.

Lemma 3.7 *Define $V_{p_n-j+1:n-j}$ as in (3.25). Assume that $k_n = n - p_n \to \infty$ and $k_n/n \to 0$ as $n \to \infty$. Then for any $\delta_n > 0$ such that $\delta_n \to \infty$ and $\delta_n = o(k_n^{1/6})$*

$$P(V_{p_n-j+1:n-j} > x) = (1 + o(1))(1 - \Phi(x)) \tag{3.14}$$

uniformly over $0 \leq x \leq \delta_n$, $1 \leq j \leq p_n - k_n$ as $n \to \infty$.

Proof. Set $\beta_{nj}(x) = \frac{p_n-j}{n-j} + \frac{((p_n-j)k_n)^{1/2}}{(n-j)^{3/2}}x = \frac{p_n-j}{n-j}(1+\frac{k_n^{1/2}}{(n-j)^{1/2}(p_n-j)^{1/2}}x)$. Then $1-\beta_{nj}(x) = \frac{n-p_n}{n-j} - \frac{((p_n-j)k_n)^{1/2}}{(n-j)^{3/2}}x = \frac{k_n}{n-j}(1 - \frac{(p_n-j)^{1/2}}{(n-j)^{1/2}k_n^{1/2}}x)$, and the density function of $V_{p_n-j+1:n-j}$ is given by

$$\begin{aligned} h_j(x) &= \frac{((p_n-j)k_n)^{1/2}}{(n-j)^{3/2}} f_{p_n-j+1:n-j}(\beta_{nj}(x)) \\ &= \frac{((p_n-j)k_n)^{1/2}}{(n-j)^{3/2}} \frac{(n-j)!}{(p_n-j)!(k_n-1)!}\beta_{nj}(x)^{p_n-j}(1-\beta_{nj}(x))^{k_n-1} \\ &= \frac{(p_n-j)^{p_n-j+1/2}k_n^{k_n+1/2}}{(n-j)^{n-j+1/2}} \frac{(n-j)!}{(p_n-j)!k_n!}(1 - \frac{(p_n-j)^{1/2}}{(n-j)^{1/2}k_n^{1/2}}x)^{-1} \qquad (3.15) \\ &\quad \times (1 + \frac{k_n^{1/2}}{(n-j)^{1/2}(p_n-j)^{1/2}}x)^{p_n-j}(1 - \frac{(p_n-j)^{1/2}}{(n-j)^{1/2}k_n^{1/2}}x)^{k_n}. \qquad (3.16) \end{aligned}$$

To estimate $h_j(x)$, we need Stirling's formula:

$$j! = j^{j+1/2}e^{-j+\varepsilon(j)}\sqrt{2\pi}, \text{ where } \frac{1}{12j+1} < \varepsilon(j) < \frac{1}{12j} \tag{3.17}$$

and Taylor's expansion: $1 - t = \exp(\log(1-t)) = \exp(-t - \frac{1}{2}t^2 + O(t^3))$ as $t \to 0$. By applying Stirling's formula to $k_n!$, $(p_n-j)!$ and $(n-j)!$, the product in (3.15) is equal to $\frac{1+o(1)}{\sqrt{2\pi}}(1+o(1))$ for $|x| = o(k_n^{1/6})$ uniformly over $1 \leq j \leq p_n - k_n$ as $n \to \infty$. By applying Taylor's expansion to $(1 + \frac{k_n^{1/2}}{(n-j)^{1/2}(p_n-j)^{1/2}}x)^{p_n-j}$ and $(1 - \frac{(p_n-j)^{1/2}}{(n-j)^{1/2}k_n^{1/2}}x)^{k_n}$, the product in

(3.16) is equal to

$$\begin{aligned}
& \exp\Big(\frac{(p_n-j)k_n^{1/2}x}{(n-j)^{1/2}(p_n-j)^{1/2}} - \frac{1}{2}\frac{(p_n-j)k_nx^2}{(n-j)(p_n-j)} + O(\frac{(p_n-j)k_n^{3/2}|x^3|}{(n-j)^{3/2}(p_n-j)^{3/2}})\Big) \\
& \times\exp\Big(-\frac{k_n(p_n-j)^{1/2}x}{(n-j)^{1/2}k_n^{1/2}} - \frac{1}{2}\frac{k_n(p_n-j)x^2}{(n-j)k_n} + O(\frac{k_n(p_n-j)^{3/2}|x|^3}{(n-j)^{3/2}k_n^{3/2}})\Big) \\
= & \exp\Big(\frac{(p_n-j)^{1/2}k_n^{1/2}x}{(n-j)^{1/2}} - \frac{1}{2}\frac{k_nx^2}{(n-j)} + O(\frac{k_n^{3/2}|x^3|}{(n-j)^{3/2}(p_n-j)^{1/2}})\Big) \\
& \times\exp\Big(-\frac{k_n^{1/2}(p_n-j)^{1/2}x}{(n-j)^{1/2}} - \frac{1}{2}\frac{(p_n-j)x^2}{(n-j)} + O(\frac{(p_n-j)^{3/2}|x|^3}{(n-j)^{3/2}k_n^{1/2}})\Big) \\
= & \exp\Big(-\frac{x^2}{2} + O((\frac{1}{k_n^{1/2}} + \frac{1}{(p_n-j)^{1/2}})|x|^3)\Big) \\
= & \exp\Big(-\frac{x^2}{2} + o(1)\Big)
\end{aligned}$$

for $|x| = o(k_n^{1/6})$ uniformly over $1 \le j \le p_n - k_n$ as $n \to \infty$. Therefore, we have

$$h_j(x) = \frac{1+o(1)}{\sqrt{2\pi}}\exp(-\frac{1}{2}x^2), \quad |x| = o(k_n^{1/6}) \tag{3.18}$$

uniformly over $1 \le j \le p_n - k_n$ as $n \to \infty$. Next we will give an estimate of the upper bound of $h_j(x)$ for large x. Note that $\beta_{nj}(x) < 1$ if and only if $x < \frac{k_n^{1/2}(p_n-j)^{1/2}}{(n-j)^{1/2}} = O(k_n^{1/2})$ uniformly over $1 \le j \le p_n - k_n$ and thus $\frac{k_n^{1/2}x}{(n-j)^{1/2}(p_n-j)^{1/2}} \le O(\frac{k_n^{1/2}}{n^{1/2}}) \to 0$ uniformly over $0 < x < \frac{k_n^{1/2}(p_n-j)^{1/2}}{(n-j)^{1/2}}$, $1 \le j \le p_n - k_n$ as $n \to \infty$. Now by applying Taylor's expansion to $(1 + \frac{k_n^{1/2}}{(n-j)^{1/2}(p_n-j)^{1/2}}x)^{p_n-j}$ and inequality

$$1 - t \le \exp(-t - \frac{1}{2}t^2), \quad t \in (0,1)$$

to $(1 - \frac{(p_n-j)^{1/2}}{(n-j)^{1/2}k_n^{1/2}}x)^{k_n}$ we get

$$h_j(x) \le \frac{1+o(1)}{\sqrt{2\pi}}\exp(-\frac{1}{2}x^2), \quad 0 < x < \frac{k_n^{1/2}(p_n-j)^{1/2}}{(n-j)^{1/2}} \tag{3.19}$$

uniformly over $1 \le j \le p_n - k_n$ as $n \to \infty$.

Assume that $0 \le x \le \delta_n$. From (3.19) we have

$$\begin{aligned}
P(V_{p_n-j+1:n-j} > x) &= \int_x^{\frac{k_n^{1/2}(p_n-j)^{1/2}}{(n-j)^{1/2}}} h_j(t)dt \\
&\le (1+o(1))\int_x^{\frac{k_n^{1/2}(p_n-j)^{1/2}}{(n-j)^{1/2}}} \phi(t)dt \\
&\le (1+o(1))(1-\Phi(x))
\end{aligned}$$

uniformly over $0 \le x \le \delta_n$, $1 \le j \le p_n - k_n$ as $n \to \infty$. Therefore, to complete the proof of the lemma, it suffices to show that

$$P(V_{p_n-j+1:n-j} > x) \ge (1+o(1))(1-\Phi(x)) \tag{3.20}$$

uniformly over $0 \le x \le \delta_n$, $1 \le j \le p_n - k_n$ as $n \to \infty$.

From (3.27), we see that

$$\Phi(y) - \Phi(x) = (1+o(1))(1-\Phi(x)), \quad 0 \le x = o(y)$$

uniformly if $y \to \infty$. Since $(\delta_n k_n^{1/6})^{1/2} = o(k_n^{1/6})$, by using (3.18) we have

$$\begin{aligned} P(V_{p_n-j+1:n-j} > x) &\ge \int_x^{(\delta_n k_n^{1/6})^{1/2}} h_j(t)dt \\ &= (1+o(1))\int_x^{(\delta_n k_n^{1/6})^{1/2}} \phi(t)dt \\ &= (1+o(1))(\Phi((\delta_n k_n^{1/6})^{1/2}) - \Phi(x)) \\ &= (1+o(1))(1-\Phi(x)), \end{aligned}$$

proving (3.20). This completes the proof of the lemma. ■

3.2 Proofs of the Theorems

Proof of Theorem 2. We need to prove

$$\frac{1}{B_n}(\max_{1\le j\le p_n} |z_j| - A_n) \xrightarrow{d} \Lambda, \tag{3.21}$$

where $A_n = c_n + \frac{1}{2}(1-c_n^2)^{1/2}(n-1)^{-1/2}a_n$, $B_n = \frac{1}{2}(1-c_n^2)^{1/2}(n-1)^{-1/2}b_n$,

$$c_n = \Big(\frac{p_n-1}{n-1}\Big)^{1/2}, \quad b_n = b\Big(\frac{nc_n^2}{1-c_n^2}\Big), \quad a_n = a\Big(\frac{nc_n^2}{1-c_n^2}\Big)$$

with $a(x) = (\log x)^{1/2} - (\log x)^{-1/2}\log(\sqrt{2\pi}\log x)$ and $b(x) = (\log x)^{-1/2}$ for $x > 3$.

Fix $x \in \mathbb{R}$ and set $t_n = t_n(x) = c_n^2 + c_n(1-c_n^2)^{1/2}(n-1)^{-1/2}(a_n + b_n x)$. For each $n \ge 2$, define z_{nj} as in (3.7), that is, $z_{nj} = 1 - F_{p_n-j+1:n-j}(t_n(x))$ for $1 \le j \le p_n$.

Since Y_{nj}^2 and $U_{p_n-j+1:n-j}$ are identically distributed, we have

$$P\Big(\max_{1\le j\le r} Y_{nj}^2 \le t_n(x)\Big) = \prod_{j=1}^r P(Y_{nj}^2 \le t_n(x)) = \prod_{j=1}^r (1 - z_{nj}) \tag{3.22}$$

for any $1 \le r \le p_n$.

Part 1. First we show (3.21) under condition (2.1). We will prove that

$$\lim_{n\to\infty} P\Big(\max_{1\le j\le p_n} Y_{nj}^2 \le t_n(x)\Big) = \exp(-e^{-x}). \tag{3.23}$$

Let $j_n = [p_n^{5/8}]$, the integer part of $p_n^{5/8}$. For $1 \le j \le j_n$, define

$$u_{nj} = \frac{(n-j)^{3/2}}{((p_n-j)k_n)^{1/2}}\left(t_n(x) - \frac{p_n-j}{n-j}\right).$$

Meanwhile, we rewrite

$$t_n(x) = \frac{p_n-1}{n-1} + \frac{((p_n-1)k_n)^{1/2}}{(n-1)^{3/2}}(a_n+b_nx).$$

Then we see that uniformly over $1 \le j \le j_n$,

$$\begin{aligned}
u_{nj} &= \left(\frac{p_n-1}{n-1} - \frac{p_n-j}{n-j}\right)\cdot\frac{(n-j)^{3/2}}{((p_n-j)k_n)^{1/2}} \\
&\qquad\qquad + \left(\frac{n-j}{n-1}\right)^{3/2}\cdot\left(\frac{p_n-1}{p_n-j}\right)^{1/2}(a_n+b_nx) \\
&= \left(\frac{p_n-j}{p_n-1}\right)^{-1/2}\cdot\left(\frac{n-j}{n-1}\right)^{1/2}\cdot\left(\frac{n-p_n}{p_n-1}\right)^{1/2}\cdot\left(\frac{n-1}{n}\right)^{-1/2}\cdot\frac{j-1}{n^{1/2}} \\
&\qquad\qquad + \left(\frac{n-j}{n-1}\right)^{3/2}\cdot\left(\frac{p_n-j}{p_n-1}\right)^{-1/2}(a_n+b_nx).
\end{aligned}$$

Now, $\frac{n-p_n}{p_n-1} = \frac{1-c_n^2}{c_n^2} \sim \frac{n}{p_n}$. Also, given $\tau \in \mathbb{R}$, trivially $\left(\frac{p_n-j}{p_n-1}\right)^{\tau} = 1 + O(\frac{j-1}{p_n})$ and $\left(\frac{n-j}{n-1}\right)^{\tau} = 1 + O(\frac{j-1}{n})$ uniformly for all $1 \le j \le j_n$. Since $a_n \sim (\log p_n)^{1/2}$ and $b_n \sim (\log p_n)^{-1/2}$, we have

$$\begin{aligned}
u_{nj} &= \frac{(1-c_n^2)^{1/2}}{n^{1/2}c_n}(j-1)(1+o(1)) + a_n + b_nx + O\Big(\frac{(j-1)(\log p_n)^{1/2}}{p_n}\Big) \\
&= \frac{(1-c_n^2)^{1/2}}{n^{1/2}c_n}(j-1)(1+o(1)) + a_n + b_nx \qquad (3.24)
\end{aligned}$$

uniformly for all $1 \le j \le j_n$ as $n \to \infty$.

In Lemma 3.6, take $r = n-j$ and $k = n-p_n$ to have

$$\sup_{B\in\mathcal{B}}\left|P(V_{p_n-j+1:n-j} \in B) - \int_B (1+l_1(t)+l_2(t))\phi(t)dt\right| = O(\frac{1}{p_n^{3/2}})$$

uniformly over $1 \le j \le j_n$ as $n \to \infty$, where

$$V_{p_n-j+1:n-j} = \frac{(n-j)^{3/2}}{((p_n-j)(n-p_n))^{1/2}}\Big(U_{p_n-j+1:n-j} - \frac{p_n-j}{n-j}\Big) \qquad (3.25)$$

and where, for $i = 1, 2$, $l_i(t)$ is a polynomial in t of degree $\le 3i$, depending on n, and all of its coefficients are of order $O((1/p_n)^{i/2})$ uniformly over $1 \le j \le j_n$ as $n \to \infty$. Now, by taking $B = (u_{nj}, \infty)$ we obtain

$$z_{nj} = P(V_{p_n-j+1:n-j} > u_{nj}) = \int_{u_{nj}}^{\infty}(1+l_1(t)+l_2(t))\phi(t)dt + O(p_n^{-3/2}) \qquad (3.26)$$

uniformly for $1 \le j \le j_n$ as $n \to \infty$. From L'Hospital's rule, we have that for any $r \ge 0$

$$\int_x^\infty t^r \phi(t)dt \sim x^{r-1}\phi(x) \qquad \text{as } x \to \infty. \tag{3.27}$$

Since $\min_{1 \le j \le j_n} u_{nj} \to \infty$ as $n \to \infty$ by (3.24), it follows from (3.27) that

$$\int_{u_{nj}}^\infty t^r \phi(t)dt \sim (u_{nj})^{r-1}\phi(u_{nj})$$

holds uniformly over $1 \le j \le j_n$. Furthermore, since the coefficients of $l_i(t)$ are uniformly bounded by $O((1/p_n)^{i/2})$ for $i = 1, 2$, we have

$$\begin{aligned} & \sum_{j=1}^{j_n} \int_{u_{nj}}^\infty (1 + l_1(t) + l_2(t))\phi(t)dt \qquad (3.28) \\ = \quad & (1 + o(1)) \sum_{j=1}^{j_n} \frac{\phi(u_{nj})}{u_{nj}} + O(\frac{1}{p_n^{1/2}}) \sum_{j=1}^{j_n} u_{nj}^3 \frac{\phi(u_{nj})}{u_{nj}} + O(\frac{1}{p_n}) \sum_{j=1}^{j_n} u_{nj}^6 \frac{\phi(u_{nj})}{u_{nj}}. \end{aligned}$$

In Lemma 3.5, by taking $x_n = nc_n^2/(1-c_n^2)$ and and define $c_{n,j}$ such that $u_{nj} = (j-1)c_{n,j} + a(x_n) + b(x_n)x$ for $1 \le j \le j_n$ with $c_{n,1} = x_n^{-1/2}$. It follows from (3.24) that $c_{n,j} = x_n^{-1/2}(1 + o(1))$ uniformly over $1 \le j \le j_n$ as $n \to \infty$, which implies $\lim_{n\to\infty} \max_{1\le j\le j_n} |c_{n,j} x_n^{1/2} - 1| = 0$. Then we get

$$\lim_{n\to\infty} \sum_{j=1}^{j_n} \frac{\phi(u_{nj})}{u_{nj}} = e^{-x}. \tag{3.29}$$

We will show that the second term and the third term on the line below (3.28) converge to zero as $n \to \infty$. By noting that $\frac{(1-c_n^2)^{1/2}}{n^{1/2}c_n} \sim p_n^{-1/2}$ we have

$$u_{nj}^3 = O(\frac{j_n^3}{p_n^{3/2}}) = O(p_n^{3/8}) \tag{3.30}$$

uniformly over $1 \le j \le j_n$. Thus, it follows from (3.13) that

$$O(\frac{1}{p_n^{1/2}}) \sum_{j=1}^{j_n} u_{nj}^3 \frac{\phi(u_{nj})}{u_{nj}} = O(\frac{1}{p_n^{1/8}}) \sum_{j=1}^{j_n} \frac{\phi(u_{nj})}{u_{nj}} = O(\frac{1}{p_n^{1/8}}) \to 0$$

as $n \to \infty$. Similarly, we have

$$O(\frac{1}{p_n}) \sum_{j=1}^{j_n} u_{nj}^6 \frac{\phi(u_{nj})}{u_{nj}} = O(\frac{1}{p_n^{1/4}}) \to 0$$

as $n \to \infty$. Therefore, by combining (3.28), (3.26) and (3.29) we get

$$\sum_{j=1}^{j_n} z_{nj} \to e^{-x} \quad \text{as } n \to \infty. \tag{3.31}$$

It follows from (3.24) that

$$u_{nj_n} \geq \frac{(j_n-1)(1+o(1))}{p_n^{1/2}} \geq \frac{p_n^{1/8}}{2}$$

for all large n. Then we estimate z_{nj_n} by using (3.26) with $j = j_n$ and (3.30)

$$\begin{aligned} z_{nj_n} &= \Big(\frac{1+o(1)}{u_{nj_n}} + O(\frac{u_{nj_n}^2}{p_n^{1/2}}) + O(\frac{u_{nj_n}^5}{p_n})\Big)\phi(u_{nj_n}) + O(\frac{1}{p_n^{3/2}}) \\ &= O(\exp(-\frac{1}{2}u_{nj_n}^2)) + O(\frac{1}{p_n^{3/2}}) \\ &= O(\exp(-\frac{1}{8}p_n^{1/4})) + O(\frac{1}{p_n^{3/2}}) \\ &= O(\frac{1}{p_n^{3/2}}) \end{aligned}$$

as $n \to \infty$. From (3.8) we have $\sum_{j=j_n+1}^{p_n} z_{nj} \leq p_n z_{nj_n} = O(p_n^{-1/2}) \to 0$ as $n \to \infty$, which together with (3.31) yields

$$\sum_{j=1}^{p_n} z_{nj} \to e^{-x} \qquad \text{as } n \to \infty.$$

We can also prove from (3.26) that $z_{n1} \to 0$ as $n \to \infty$. In view of (3.22) and Lemma 3.1 we conclude (3.23), ie.,

$$\frac{\max_{1\leq j\leq p_n} Y_{nj}^2 - \beta_n}{\alpha_n} \xrightarrow{d} \Lambda,$$

where $\alpha_n = c_n(1-c_n^2)^{1/2}(n-1)^{-1/2}b_n$ and $\beta_n = c_n^2 + c_n(1-c_n^2)^{1/2}(n-1)^{-1/2}a_n$. Since

$$\frac{\alpha_n}{\beta_n} \leq \frac{c_n(1-c_n^2)^{1/2}(n-1)^{-1/2}b_n}{c_n(1-c_n^2)^{1/2}(n-1)^{-1/2}a_n} = \frac{b_n}{a_n} \to 0$$

as $n \to \infty$, we can apply Lemma 3.4 and get that

$$\Lambda_n := \frac{\max_{1\leq j\leq p_n} Y_{nj} - \beta_n^{1/2}}{\alpha_n/(2\beta_n^{1/2})} \xrightarrow{d} \Lambda. \tag{3.32}$$

Recall that $A_n = c_n + \frac{1}{2}(1-c_n^2)^{1/2}(n-1)^{-1/2}a_n = c_n(1+o(1))$ and $B_n = \frac{1}{2}(1-c_n^2)^{1/2}(n-1)^{-1/2}b_n \sim \frac{1}{2}(n-1)^{-1/2}(\log p_n)^{-1/2}$. Then

$$\begin{aligned} \beta_n^{1/2} &= c_n(1 + \frac{(1-c_n^2)^{1/2}a_n}{(n-1)^{1/2}c_n})^{1/2} \\ &= c_n(1 + \frac{(1-c_n^2)^{1/2}a_n}{2(n-1)^{1/2}c_n} + O(\frac{(1-c_n^2)a_n^2}{(n-1)c_n^2})) \\ &= A_n + O(\frac{\log p_n}{(np_n)^{1/2}}) \\ &= A_n + o(B_n) \end{aligned}$$

and

$$\frac{\alpha_n}{2\beta_n^{1/2}} = B_n(1+o(1)),$$

which, together with (3.32), yield

$$\begin{aligned}\frac{\max_{1\le j\le p_n} Y_{nj} - A_n}{B_n} &= \frac{\Lambda_n\alpha_n/(2\beta_n^{1/2}) + \beta_n^{1/2} - A_n}{B_n} \\ &= \frac{\Lambda_n B_n(1+o(1)) + o(B_n)}{B_n} \\ &= (1+o(1))\Lambda_n + o(1) \\ &\stackrel{d}{\to} \Lambda\end{aligned}$$

ie., (3.21) holds. The proof of *Part 1* is completed.

Part 2. We will show (3.21) under condition (2.2). First, it follows from condition (2.2) that

$$\frac{nc_n^2}{1-c_n^2} = \frac{n(p_n-1)}{n-p_n} \sim \frac{n^2}{k_n} \to \infty$$

as $n\to\infty$. Noting that $n \le n^2/k_n \le n^2$, we get that

$$a_n = a\Big(\frac{nc_n^2}{1-c_n^2}\Big) \sim \Big(\log\Big(\frac{nc_n^2}{1-c_n^2}\Big)\Big)^{1/2}$$

is of order $(\log n)^{1/2}$ as $n\to\infty$.

Use the same notation as in Part 1. Recall that $p_n/n \to 1$, $k_n = n - p_n = o(n)$ and $\log n = o(k_n^{1/3})$ as $n\to\infty$. In order to use both Lemmas 3.5 and 3.7, we take $x_n = \frac{nc_n^2}{1-c_n^2}$. Define $j_n = [5(\log n)^{1/2}\sqrt{x_n}] + 1$. Then $j_n \sim \frac{5n(\log n)^{1/2}}{\sqrt{k_n}} = o(n)$, which implies $1 \le j_n \le p_n - k_n$ for all large n. Define $c_{n,j}$ for $1\le j\le j_n$ in the same way as in *Part 1*. Similar to the proof of (3.24) we can show that

$$u_{nj} = \frac{(1-c_n^2)^{1/2}}{n^{1/2}c_n}(j-1)(1+o(1)) + a_n + b_n x \tag{3.33}$$

uniformly for $1\le j\le j_n$ as $n\to\infty$. Then $u_{nj} = O((\log n)^{1/2}) = o(k_n^{1/6})$ uniformly for $1\le j\le j_n$ as $n\to\infty$. We can also verify that all conditions in Lemma 3.5 are satisfied. Thus, from (3.14) and (3.12) we have

$$\sum_{j=1}^{j_n} z_{nj} = (1+o(1))\sum_{j=1}^{j_n}(1-\Phi(u_{nj})) \to e^{-x} \tag{3.34}$$

as $n\to\infty$.

Next, we will show that

$$\lim_{n\to\infty}\sum_{j=j_n+1}^{p_n} z_{nj} = 0. \tag{3.35}$$

Note that (3.14) holds uniformly over $1 \le j \le p_n - k_n$ and $u_{nj_n} \ge 4(\log n)^{1/2}$ for all large n. By employing (3.14) with $j = j_n$ and $x = 4(\log n)^{1/2}$ and using equation (3.8) and Lemma 3.7 we have

$$\begin{aligned}
\sum_{j=j_n+1}^{p_n} z_{nj} &\le n z_{nj_n} \\
&\le nP(V_{p_n-j_n+1:n-j_n} > u_{nj_n}) \\
&\le nP(V_{p_n-j_n+1:n-j_n} \ge 4(\log n)^{1/2}) \\
&= (1+o(1))n(1-\Phi(4(\log n)^{1/2})) \\
&= O(\frac{1}{n}) \\
&\to 0.
\end{aligned}$$

Thus, we obtain that $\sum_{j=1}^{p_n} z_{nj} \to e^{-x}$ for any x. Then equation (3.23) follows from equation (3.22) and Lemma 3.1. The rest of the proof will follow from the same lines in the proof of the first part. Again Lemma 3.4 will be used. The details are omitted. ■

Proof of Theorem 3. Recall that a_n is given by

$$\frac{1}{(k_n-1)!}\int_0^{a_n} t^{k_n-1}e^{-t}dt = \frac{k_n}{n}. \tag{3.36}$$

By using integration by parts, we have

$$y^{k_n}e^{-y} = k_n\int_0^y t^{k_n-1}e^{-t}dt - \int_0^y t^{k_n}e^{-t}dt \ge k_n\int_0^y t^{k_n-1}e^{-t}dt - y\int_0^y t^{k_n-1}e^{-t}dt,$$

which implies

$$\frac{1}{k_n}y^{k_n}e^{-y} \le \int_0^y t^{k_n-1}e^{-t}dt \le \frac{1}{k_n-y}y^{k_n}e^{-y}, \quad 0 \le y < k_n. \tag{3.37}$$

By using Stirling's formula (3.17), we have under condition (2.3) that

$$y_n := (\frac{k_n!k_n}{nk_n^{k_n}})^{1/k_n} = \exp(\frac{3}{2}\frac{\log k_n}{k_n} - 1 + \frac{\varepsilon(k_n)+\log\sqrt{2\pi}}{k_n})\exp(-\frac{\log n}{k_n}) \to 0$$

Since te^{-t} is strictly increasing in $(0,1)$, for all large n such that $y_n < 1$ define ε_n as the unique solution to $te^{-t} = y_n$ in $(0,1)$, that is, $\varepsilon_n e^{-\varepsilon_n} = y_n$, which implies that $\varepsilon_n \to 0$ and $\varepsilon_n \sim y_n$ as $n \to \infty$ and

$$\frac{1}{k_n!}(k_n\varepsilon_n)^{k_n}e^{-k_n\varepsilon_n} = \frac{k_n}{n}$$

for all large n. Then it follows from the the first inequality in (3.37) that

$$\frac{1}{(k_n-1)!}\int_0^{k_n\varepsilon_n} t^{k_n-1}e^{-t}dt \ge \frac{k_n}{n}$$

for all large n, which together with (3.36) implies that $a_n \le k_n\varepsilon_n$ for all large n,and thus $a_n = o(k_n)$ as $n \to \infty$. By plugging $y = a_n$ in (3.37) and using $a_n \le k_n\varepsilon_n$ for large n we conclude

$$\int_0^{a_n} t^{k_n-1}e^{-t}dt = \frac{1}{k_n}a_n^{k_n}e^{-a_n}(1+O(\varepsilon_n))$$

which implies

$$\frac{k_n}{n} = \frac{1}{(k_n-1)!}\int_0^{a_n} t^{k_n-1}e^{-t}dt = \frac{1}{k_n!}a_n^{k_n}e^{-a_n}(1+O(\varepsilon_n))$$

as $n \to \infty$, and consequently

$$\frac{n}{k_n!k_n}a_n^{k_n}e^{-a_n} = 1+o(1) \quad \text{as } n \to \infty. \tag{3.38}$$

Define z_{nj} as in (3.7) with $t_n = t_n(x) = 1 - \frac{a_n}{n}(1-\frac{x}{k_n})$ for any fixed x. Then $1 - t_n = \frac{a_n}{n}(1-\frac{x}{k_n}) = o(\frac{k_n}{n}) = o(\frac{\log n}{n})$ as $n \to \infty$, where we have used the fact that $k_n = n-p_n \to \infty$ and $k_n = o(\log n)$ from (2.3). This implies $n(1-t_n)^2 \to 0$ as $n \to \infty$.

It is easy to verify the following expression

$$1-t = e^{-t-d(t)}, \quad 0 \le t \le 1/2$$

where $0 \le d(t) \le t^2$ for $0 \le t \le 1/2$. Then

$$\begin{aligned}
\int_{t_n}^1 t^{p_n-j}(1-t)^{k_n-1}dt &= \int_0^{1-t_n}(1-t)^{p_n-j}t^{k_n-1}dt \\
&= (1+o(1))\int_0^{1-t_n} t^{k_n-1}e^{-(p_n-j)t}dt \\
&= (1+o(1))\int_0^{1-t_n} t^{k_n-1}e^{-(n-j)t}e^{k_nt}dt \\
&= (1+o(1))\int_0^{1-t_n} t^{k_n-1}e^{-(n-j)t}dt \\
&= \frac{1+o(1)}{(n-j)^{k_n}}\int_0^{(1-t_n)(n-j)} t^{k_n-1}e^{-t}dt.
\end{aligned}$$

Furthermore, by using Stirling's formula (3.17), we get

$$\begin{aligned}
\frac{(n-j)!}{(p_n-j)!} &= \frac{(n-j)^{n-j+1/2}e^{-(n-j)+\varepsilon(n-j)}}{(p_n-j)^{p_n-j+1/2}e^{-(p_n-j)+\varepsilon(p_n-j)}} \\
&= (1+\frac{k_n}{p_n-j})^{p_n-j+1/2}(n-j)^{k_n}e^{-k_n}e^{\varepsilon(p_n-j)-\varepsilon(n-j)} \\
&= (1+o(1))(n-j)^{k_n}
\end{aligned}$$

uniformly over $1 \le j \le j_n$, where $j_n := p_n - k_n^3$. Then from (3.9) we obtain that

$$z_{nj} = \frac{1+o(1)}{(k_n-1)!}\int_0^{(1-t_n)(n-j)} t^{k_n-1}e^{-t}dt \tag{3.39}$$

uniformly over $1 \le j \le j_n$.

Note that $n(1-t_n) = o(k_n)$. Then it follows from (3.39) and (3.37) that

$$\begin{aligned} z_{nj} &= \frac{1+o(1)}{k_n!}(n-j)^{k_n}(1-t_n)^{k_n}e^{-(n-j)(1-t_n)} \\ &= \frac{1+o(1)}{1-t_n}\int_{(n-j)(1-t_n)}^{(n-j+1)(1-t_n)} t^{k_n}e^{-t}dt \end{aligned}$$

uniformly over $1 \le j \le j_n$, and thus

$$\begin{aligned} \sum_{j=1}^{j_n} z_{nj} &= \frac{1+o(1)}{k_n!}\sum_{j=1}^{j_n}(n-j)^{k_n}(1-t_n)^{k_n}e^{-(n-j)(1-t_n)} \\ &= \frac{1+o(1)}{k_n!}\sum_{j=1}^{j_n}\frac{1}{1-t_n}\int_{(n-j)(1-t_n)}^{(n-j+1)(1-t_n)} t^{k_n}e^{-t}dt \\ &= \frac{1+o(1)}{k_n!}\frac{1}{1-t_n}\int_{(n-j_n)(1-t_n)}^{n(1-t_n)} t^{k_n}e^{-t}dt \\ &= \frac{1+o(1)}{k_n!}\frac{1}{1-t_n}\Big(\int_0^{n(1-t_n)} t^{k_n}e^{-t}dt - \int_0^{k_n^3(1-t_n)} t^{k_n}e^{-t}dt\Big). \end{aligned}$$

The second integral above is dominated by the first one since $t^{k_n}e^{-t}$ is increasing over $(0, k_n)$ and $k_n^3(1-t_n)/(n(1-t_n)) = k_n^3/n \to 0$ as $n \to \infty$. Therefore, in view of (3.37) and (3.38) we have

$$\begin{aligned} \sum_{j=1}^{j_n} z_{nj} &= \frac{1+o(1)}{k_n!}\frac{1}{1-t_n}\int_0^{n(1-t_n)} t^{k_n}e^{-t}dt \\ &= \frac{1+o(1)}{(k_n+1)!}\frac{1}{1-t_n}(n(1-t_n))^{k_n+1}e^{-n(1-t_n)} \\ &= \frac{(1+o(1))n}{(k_n+1)!}(n(1-t_n))^{k_n}e^{-n(1-t_n)} \\ &= \frac{(1+o(1))n}{k_n!k_n}(n(1-t_n))^{k_n}e^{-n(1-t_n)} \\ &= \frac{(1+o(1))n}{k_n!k_n}a_n^{k_n}(1-\frac{x}{k_n})^{k_n}e^{-a_n+O(\frac{a_n}{k_n})} \\ &= \frac{(1+o(1))n}{k_n!k_n}a_n^{k_n}e^{-a_n}e^{-x} \\ &\to e^{-x} \end{aligned}$$

as $n \to \infty$. Therefore, it follows from Lemma 3.2 that $\sum_{j=1}^{p_n} z_{nj} \to e^{-x}$ as $n \to \infty$. It is easy to conclude that $z_{n1} \to 0$ as $n \to \infty$ from (3.39), (3.37) and the above estimates. Accordingly, by taking $\beta_n = 1 - \frac{a_n}{n}$ and $\alpha_n = \frac{a_n}{nk_n}$ with $G(x) = \Lambda(x)$ in Lemmas 3.3 and

3.4 we conclude that

$$\begin{aligned}
\frac{\max_{1\le j\le p_n}|z_j|-(1-a_n)^{1/2}}{a_n/(2nk_n)} &= \frac{1}{(1-a_n)^{1/2}}\frac{\max_{1\le j\le p_n}|z_j|-(1-a_n)^{1/2}}{a_n/(nk_n2(1-a_n)^{1/2})}\\
&= (1+o(1))\frac{\max_{1\le j\le p_n}|z_j|-(1-a_n)^{1/2}}{a_n/(nk_n2(1-a_n)^{1/2})}\\
&\stackrel{d}{\to} \Lambda
\end{aligned}$$

as $n\to\infty$. This completes the proof of the theorem. ■

Proof of Theorem 4. We first show that

$$\frac{n^{1+1/k}}{((k+1)!)^{1/k}}(\max_{1\le j\le p_n}|z_j|^2-1)\stackrel{d}{\to} W_k \tag{3.40}$$

Fix $x<0$. Let $t_n=t_n(x)=1+\frac{((k+1)!)^{1/k}}{n^{1+1/k}}x$. Then $t_n\in(0,1)$ for all large n. Since $n(1-t_n)\to 0$ as $n\to\infty$, we have

$$(1-t)^{p_n-j}=1+o(1) \quad \text{uniformly over } 0\le t\le 1-t_n,\ 1\le j\le p_n.$$

Therefore, we have from (3.7) and (3.9) that

$$\begin{aligned}
z_{nj}=1-F_{p_n-j+1:n-j}(t_n) &= \frac{(n-j)!}{(p_n-j)!(k-1)!}\int_0^{1-t_n}(1-t)^{p_n-j}t^{k-1}dt\\
&= \frac{(n-j)!}{(p_n-j)!(k-1)!}(1+o(1))\int_0^{1-t_n}t^{k-1}dt\\
&= \frac{(n-j)!}{(p_n-j)!k!}(1+o(1))(1-t_n)^k\\
&= \frac{(n-j)!}{(p_n-j)!}(1+o(1))\frac{(k+1)(-x)^k}{n^{k+1}}
\end{aligned}$$

uniformly over $1\le j\le p_n=n-k$. Since

$$\frac{(n-j)!}{(p_n-j)!}\le n^k$$

we have $\max_{1\le j\le p_n} z_{nj}=O(1/n)\to 0$ as $n\to\infty$. To complete the proof of (3.40), by using (3.5) we need to show that

$$\lim_{n\to\infty}\sum_{j=1}^{p_n}\frac{(n-j)!}{(p_n-j)!}\frac{k+1}{n^{k+1}}=1. \tag{3.41}$$

Let $\{j_n\}$ be a sequence of integers such that $j_n\to\infty$ and $j_n/n\to 0$ as $n\to\infty$. Then

$$\sum_{j=n-j_n+1}^{p_n}\frac{(n-j)!}{(p_n-j)!}\frac{k+1}{n^{k+1}}\le\frac{(k+1)j_n}{n}\to 0$$

as $n \to \infty$, and

$$\frac{(n-j)!}{(p_n-j)!} = (p_n - j + 1)^k \prod_{\ell=1}^{k} (1 + \frac{\ell - 1}{p_n - j + 1}) = (p_n - j + 1)^k (1 + o(1))$$

uniformly over $1 \le j \le n - j_n$, which implies that

$$\begin{aligned} \sum_{j=1}^{p_n - j_n} \frac{(n-j)!}{(p_n-j)!} \frac{k+1}{n^{k+1}} &= \frac{(1+o(1))(k+1)}{n} \sum_{j=1}^{p_n - j_n} (\frac{p_n - j + 1}{n})^k \\ &= \frac{(1+o(1))(k+1)}{n} \sum_{j=j_n+1}^{p_n} (\frac{j}{n})^k \\ &\to (k+1) \int_0^1 t^k dt = 1 \end{aligned}$$

as $n \to \infty$. This proves (3.41) and thus we obtain (3.40).

Finally, the theorem follows from Lemma 3.4 with $\alpha_n = \frac{((k+1)!)^{1/k}}{n^{1+1/k}}$ and $\beta_n = 1$. This completes the proof. ■

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