Limiting Spectral Radii of Circular Unitary Matrices Under Light Truncation

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Abstract

Consider a truncated circular unitary matrix which is a p_n by p_n submatrix of an n by n circular unitary matrix after deleting the last $n - p_n$ columns and rows. Jiang and Qi [17] and Gui and Qi [13] study the limiting distributions of the maximum absolute value of the eigenvalues (known as spectral radius) of the truncated matrix. Some limiting distributions for the spectral radius for the truncated circular unitary matrix have been obtained under the following conditions: (1). p_n/n is bounded away from 0 and 1; (2). $p_n \to \infty$ and $p_n/n \to 0$ as $n \to \infty$; (3). $(n - p_n)/n \to 0$ and $(n-p_n)/(\log n)^3 \to \infty$ as $n \to \infty$; (4). $n-p_n \to \infty$ and $(n-p_n)/\log n \to 0$ as $n \to \infty$; and (5). $n-p_n = k \geq 1$ is a fixed integer. The spectral radius converges in distribution to the Gumbel distribution under the first four conditions and to a reversed Weibull distribution under the fifth condition. Apparently, the conditions above do not cover the case when $n - p_n$ is of order between $\log n$ and $(\log n)^3$. In this paper, we prove that the spectral radius converges in distribution to the Gumbel distribution as well in this case, as conjectured by Gui and Qi [13].

Keywords: Spectral radius; eigenvalue; limiting distribution; extreme value; circular unitary matrix

AMS 2010 Subject Classification: 60F99, 60G55, 60G70.

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1 Introduction

The study of large random matrices can date back to nearly a century ago, and one example is Wishart's [33] work on statistical properties for large covariance matrices. The theory of random matrices has been rapidly developed in last few decades and has found applications in heavy-nuclei atoms (Wigner [32]), number theory (Mezzadri and Snaith [23]), quantum mechanics (Mehta [22]), condensed matter physics (Forrester [11]), wireless communications (Couillet and Debbah [7]), to just mention a few.

Statistical properties of large random matrices including their empirical spectral distributions and spectral radii (the largest eigenvalues) are of particular interest in the study. For the three Hermitian matrices including Gaussian orthogonal ensemble, Gaussian unitary ensemble and Gaussian symplectic ensemble, Tracy and Widom [29, 30] show that their spectral radii converge in distribution to Tracy-Widom laws. For more consequent applications of Tracy-Widom laws, see, e.g., Baik et al. [3], Tracy and Widom [31], Johansson [19], Johnstone [20, 21], and Jiang [15]. For a non-Hermitian matrix, the largest absolute value of its eigenvalues is referred to as the spectral radius. The spectral radii for the real, complex and symplectic Ginibre ensembles are explored by Rider [25, 26] and Rider and Sinclair [27], and their limiting distributions are usually the Gumbel distributions instead of the Tracy-Widow laws.

In this paper, we are interested in the truncation of the circular unitary ensemble. The circular unitary ensemble is a random square matrix with Haar measure on the unitary group, and it is also called Haar-invariant unitary matrix. Truncations of large Haar unitary matrices are employed to describe quantum systems with absorbing boundaries (Casati et al. [4]) and have applications in optical and semiconductor superlattices (Glück et al. [12]) and quantum conductance (Forrester [10]), among many others. More references on applications can be found in Dong et al. [9].

Let $\mathbf{U}$ be an $n \times n$ circular unitary matrix. The n eigenvalues of the circular unitary matrix $\mathbf{U}$ are distributed over $\{z \in \mathcal{C} : |z| = 1\}$, where $\mathcal{C}$ is the complex plane, and their joint density function is given by

$$\frac{1}{n!(2\pi)^n} \cdot \prod_{1\le j<k\le n} |z_j - z_k|^2;$$

see, e.g., Hiai and Petz [14]. For integer p with $1 \le p < n$, partition $\mathbf{U}$ as follows

$$\mathbf{U} = \begin{pmatrix} \mathbf{A} & \mathbf{C}^* \\ \mathbf{B} & \mathbf{D} \end{pmatrix}$$

where $\mathbf{A}$, as a truncation of $\mathbf{U}$, is a $p \times p$ submatrix. Let $\mathbf{z}_1, \cdots, \mathbf{z}_p$ be the p eigenvalues of $\mathbf{A}$. According to Życzkowski and Sommers [34], their density function is

$$C \cdot \prod_{1 \le j < k \le p} |z_j - z_k|^2 \prod_{j=1}^{p} (1 - |z_j|^2)^{n-p-1} I(|z_j| < 1) \tag{1.1}$$

where C is a constant, depending on both n and p such that the above function is a probability density.

In this paper we assume that $p = p_n$ depends on n and $\lim_{n\to\infty} p_n = \infty$.

Set $c = \lim_{n\to\infty}(p_n/n)$. Życzkowski and Sommers [34] prove that the empirical spectral distribution of $\mathbf{z}_i$'s converges to the distribution with density proportional to $\frac{1}{(1-|z|^2)^2}$ for $|z| \le c$ if $c \in (0, 1)$. Dong et al. [9] show that the empirical spectral distribution goes to the circular law and the arc law as $c = 0$ and $c = 1$, respectively. For more work, see also Diaconis and Evans [8] and Jiang [15, 16].

Two recent papers by Jiang and Qi [17] and Gui and Qi [13] study the limiting distributions of the spectral radius $\max_{1\le j\le p} |\mathbf{z}_j|$ for the truncated circular unitary ensemble. Jiang and Qi [17] have proved that the spectral radius $\max_{1\le j\le p} |\mathbf{z}_j|$ converges to the Gumbel distribution when the ratio p_n/n is bounded away from 0 and 1. Gui and Qi [13] further consider the case when the limit of p_n/n is 0 or 1. Since $\mathbf{A}$ is obtained by deleting last $n - p_n$ rows and columns from $\mathbf{U}$, we call the truncation is light if $\lim_{n\to\infty}(n - p_n)/n = 0$, otherwise, the truncation is heavy if $\liminf_{n\to\infty}(n - p_n)/n > 0$. The main results obtained by Jiang and Qi [17] and Gui and Qi [13] are summarized in section 2.

Obvious, from Jiang and Qi [17] and Gui and Qi [13] we observe that the limiting distribution for the spectral radius $\max_{1\le j\le p} |\mathbf{z}_j|$ depends on the truncation parameter $n - p_n$. We are interested in investigating how the limiting distribution of the spectral radius changes when the truncation parameter runs over the range $1 \le n - p_n < n$ under constrain that $\lim_{n\to\infty} p_n = \infty$. The limiting distribution for the spectral radius $\max_{1\le j\le p} |\mathbf{z}_j|$ remains unknown when the truncation parameter $n - p_n$ is of order between $\log n$ and $(\log n)^3$. Gui and Qi [13] conjecture that $\max_{1\le j\le p_n} |\mathbf{z}_j|$, after properly normalized, converges in distribution to the Gumbel distribution in this case. In this paper, we will show that this conjecture is true. This paper together with Jiang and Qi [17] and Gui and Qi [13] will put an end to the study of the limiting spectral radius for the truncated circular unitary ensemble. It is worth noting that the key approaches for the proofs in Jiang and Qi [17] and Gui and Qi [13] are no longer applicable in the aforementioned regime, and therefore, we have to use a totally different approach in this paper. More details will be provided in **Remark** 2 in section 2.

The rest of the paper is organized as follows. The main result in this paper is given in section 2 and the proofs for auxiliary lemmas and the main result will be given in section 3.

2 Main Result

Consider the $p_n \times p_n$ submatrix $\mathbf{A}$, truncated from a $n \times n$ circular unitary matrix $\mathbf{U}$ in section 1. Denote the p_n eigenvalues of $\mathbf{A}$ as $\mathbf{z}_1, \cdots, \mathbf{z}_{p_n}$ with the joint density function given by (1.1).

The limiting distribution for the spectral radius $\max_{1\le j\le p_n} |\mathbf{z}_j|$ has been obtained by Jiang and Qi [17] and Gui and Qi [13] under each of the following conditions:

$$0 < h_1 < \frac{p_n}{n} < h_2 < 1, \text{ where } h_1 \text{ and } h_2 \text{ are two constants;} \tag{2.1}$$

$$p_n \to \infty \text{ and } \frac{p_n}{n} \to 0 \text{ as } n \to \infty; \tag{2.2}$$

$$\frac{n-p_n}{(\log n)^3} \to \infty \text{ and } \frac{n-p_n}{n} \to 0 \text{ as } n \to \infty; \tag{2.3}$$

$$n-p_n \to \infty \text{ and } \frac{n-p_n}{\log n} \to 0 \text{ as } n \to \infty; \tag{2.4}$$

$$n - p_n = k \ge 1 \text{ is a fixed integer.} \tag{2.5}$$

Theorems 1 to 3 below are summarized from Jiang and Qi [17] and Gui and Qi [13]. The main contribution of the present paper is Theorem 4.

THEOREM 1 *Assume that* $\mathbf{z}_1, \cdots, \mathbf{z}_{p_n}$ *have density as in (1.1), and* $\{p_n\}$ *is a sequence of positive integers satisfying* $1 \le p_n < n$ *and*

$$p_n \to \infty \text{ and } \frac{n - p_n}{(\log n)^3} \to \infty \text{ as } n \to \infty. \tag{2.6}$$

Then $(\max_{1\le j\le p_n} |\mathbf{z}_j| - A_n)/B_n$ *converges weakly to the Gumbel distribution* $\Lambda(x) = \exp(-e^{-x})$, $x \in \mathbb{R}$, *where* $A_n = c_n + \frac{1}{2}(1-c_n^2)^{1/2}(n-1)^{-1/2}a_n$, $B_n = \frac{1}{2}(1-c_n^2)^{1/2}(n-1)^{-1/2}b_n$,

$$c_n = \Big(\frac{p_n - 1}{n-1}\Big)^{1/2}, \quad b_n = b\Big(\frac{nc_n^2}{1-c_n^2}\Big), \quad a_n = a\Big(\frac{nc_n^2}{1-c_n^2}\Big)$$

with

$$a(y) = (\log y)^{1/2} - (\log y)^{-1/2}\log(\sqrt{2\pi}\log y) \quad \text{and} \quad b(y) = (\log y)^{-1/2} \quad \text{for } y > 3.$$

THEOREM 2 *Under condition* (2.4), $(\max_{1\le j\le p_n} |\mathbf{z}_j| - A_n)/B_n$ *converges weakly to the Gumbel distribution* $\Lambda(x) = \exp(-e^{-x})$, $x \in \mathbb{R}$, *where* $A_n = (1 - a_n/n)^{1/2}$ *and* $B_n = a_n/(2nk_n)$ *with* $k_n = n - p_n$, *and* a_n *is given by*

$$\frac{1}{(k_n-1)!}\int_0^{a_n} t^{k_n-1}e^{-t}dt = \frac{k_n}{n}.$$

THEOREM 3 *Under condition* (2.5), $\frac{2n^{1+1/k}}{((k+1)!)^{1/k}}(\max_{1\le j\le p_n} |\mathbf{z}_j| - 1)$ *converges weakly to the reversed Weibull distribution* $W_k(x)$ *defined as*

$$W_k(x) = \begin{cases} \exp(-(-x)^k), & x \le 0; \\ 1, & x > 0. \end{cases}$$

Remark 1. Theorems 2 and 3 are proved in Gui and Qi [13]. Theorem 1 reduces to Theorem 2 in Jiang and Qi [17] under (2.1) and to Theorem 2 in Gui and Qi [13] under (2.2) or (2.3). Note that condition (2.6) combines conditions (2.1), (2.2) and (2.3). In fact, Theorem 1 can be concluded from Theorem 2 in Jiang and Qi [17] and Theorem 2 in Gui and Qi [13] by using subsequence arguments. A proof can be outlined as follows. Let $\{p_n\}$ be any sequence satisfying (2.6). Then for any subsequence $\{n'\}$ of positive integers, there always exists its further subsequence, say $\{n''\}$, such that one of the three conditions (2.1), (2.2) and (2.3) holds along the subsequence $\{n''\}$. By applying Theorem 2 in Jiang and Qi [17] or Theorem 2 in Gui and Qi [13], we know that Theorem 1 holds along the subsequence $\{n''\}$. This is sufficient to conclude Theorem 1 above.

When $n-p_n$ is of order between $\log n$ and $(\log n)^3$, neither of conditions from (2.1) to (2.5) holds. In this paper, we consider the following condition

$$k_n = n - p_n \to \infty \quad \text{and} \quad \frac{k_n(\log n)^3}{n} \to 0 \quad \text{as} \quad n \to \infty. \tag{2.7}$$

The range of p_n here is wide enough to cover the gap that is not considered in Jiang and Qi [17] and Gui and Qi [13].

To define the normalizing constants for $\max_{1\le j\le p_n} |\mathbf{z}_j|$, set λ_n as the solution to

$$g_n(\lambda) := \lambda - 1 - \log(\lambda) + \frac{2}{k_n}\log(1-\lambda) = \frac{1}{k_n}\log\left(\frac{n}{2\pi k_n^{3/2}}\right) \tag{2.8}$$

in $(0,1)$. We see that $g_n(\lambda)$ is decreasing in $(0,1)$ by noting

$$g_n'(\lambda) = 1 - \frac{1}{\lambda} - \frac{2k_n}{1-\lambda} < 0 \qquad \text{for } \lambda \in (0,1).$$

Since $g_n(0+) = \infty$ and $g_n(1-) = -\infty$, a unique solution to $g_n(\lambda) = c$ in $(0,1)$ exists for any constant c.

Our main contribution in the paper is the following Theorem 4, which confirms the conjecture by Gui and Qi [13].

THEOREM 4 *Under condition* (2.7), $(\max_{1\le j\le p_n} |\mathbf{z}_j| - A_n)/B_n$ *converges weakly to the Gumbel distribution* $\Lambda(x) = \exp(-e^{-x})$, $x \in \mathbb{R}$, *where*

$$A_n = \left(1 - \frac{k_n\lambda_n}{n}\right)^{1/2}, \qquad B_n = \frac{\lambda_n}{2A_n n(1-\lambda_n)}.$$

Remark 2. The eigenvalues for truncation of the circular unitary ensemble form a determinantal point process and share the property of intrinsic independence. This property is very helpful in investigating both the asymptotic distribution of the spectral radius and the empirical spectral distribution of the eigenvalues from a determinantal point process; see, e.g., Jiang and Qi [18], Chang and Qi [6], Chang, Li and Qi [5] for more work on limiting empirical spectral distributions for non-Hermitian random matrices. The proof of Theorem 4 is quite lengthy and will be split into a series of auxiliary lemmas in section 3. For the case under heavy truncation, Gui and Qi [13] and Jiang and Qi [17] employ moderate deviation principles for sum of independent random variable, but this approach does not work anymore for our case. In fact, when $k_n = n - p_n$ is of order between $\log n$ and $(\log n)^3$, we need a uniform estimate of the probability for a Gamma(k_n) random variable falling into the interval $(0, x]$, where x is between 0 and some constant c_n with $c_n < k_n$. Obviously, this is beyond the range one can apply moderate deviation principles for Gamma(k_n) since Gamma(k_n) is the sum of k_n independent Gamma(1) random variables. Instead, we obtain a fine estimate in Lemma 3.4 below for large-parameter incomplete Gamma function via a result in Temme [28]. This lemma, together with Lemmas 3.5, 3.6 and 3.7 on several estimates of functions of the solution λ_n to equation (2.8), enables us to prove Lemma 3.8 and Theorem 4. Meanwhile, this method may not be easily extended to prove the results from Jiang and Qi [17] and Gui and Qi [13] in general since approximations for some other terms will get worse if k_n is too large. Fortunately, the range of p_n under condition (2.7) is wide enough to bridge the gap in the literature.

3 Proofs

We need the following notation in our proofs. We use the symbol $C_n \sim D_n$ to denote the relationship $\lim_{n\to\infty} \frac{C_n}{D_n} = 1$. For random variables $\{X_n, n \geq 1\}$ and constants $\{a_n, n \geq 1\}$, we write $X_n = O_p(a_n)$ if $\lim_{x\to+\infty} \limsup_{n\to\infty} P(|\frac{X_n}{a_n}| \geq x) = 0$, and we write $X_n = o_p(a_n)$ if $\frac{X_n}{a_n} \to 0$ in probability. It is well known that $\frac{X_n}{a_n b_n} \to 0$ in probability as $n \to \infty$ if $X_n = O_p(a_n)$ and $\{b_n, n \geq 1\}$ is a sequence of constants with $\lim_{n\to\infty} b_n = \infty$.

As in Gui and Qi [13], assume that $\{U_i, \ i \geq 1\}$ is a sequence of independent and identically distributed (i.i.d.) random variables uniformly distributed over $(0, 1)$, and $U_{1:n} \leq U_{2:n} \leq \cdots \leq U_{n:n}$ denote the order statistics of $U_1, U_2, \cdots, U_n$ for each $n \geq 1$. Then $U_{i:n}$ has a Beta$(i, n - i + 1)$ distribution with density function given by

$$f_{i:n}(x) = \frac{n!}{(i-1)!(n-i)!} x^{i-1}(1-x)^{n-i}, \quad 0 < x < 1,$$

and its cumulative distribution function (cdf) is denoted by $F_{i:n}(x)$, $0 \leq x \leq 1$.

For each $n \geq 2$, let $\{Y_{nj}, 1 \leq j \leq p_n\}$ be independent random variables such that Y_{nj} and $(U_{p_n+1-j:n-j})^{1/2}$ have the same distribution for each j. Jiang and Qi [17] have shown that $\max_{1\leq j\leq p_n} |\mathbf{z}_j|^2$ and $\max_{1\leq j\leq p_n} Y_{nj}^2$ have the same distribution.

Next, we express each Beta random variable in terms of Gamma random variables. From equation (2.2.1) on page 12 in Ahsanullah and Nevzorov [2], we have, for each $1 \le k \le n$, $U_{k:n}$ and $\sum_{i=1}^{k} E_i / \sum_{i=1}^{n+1} E_i$ have the same distribution, where $\{E_i,\ i \ge 1\}$ is a sequence of independent random variables with the standard exponential distribution. In fact, if we assume that $\{E_{ij}, i \ge 1, j \ge 1\}$ are independent random variables with the standard exponential distribution, then $\{\sum_{i=1}^{p_n+1-j} E_{i,j} / \sum_{i=1}^{n+1-j} E_{i,j},\ 1 \le j \le p_n\}$ are independent random variables, and for each $1 \le j \le p_n$, $\sum_{i=1}^{p_n+1-j} E_{i,j} / \sum_{i=1}^{n+1-j} E_{i,j}$ and $U_{p_n+1-j:n-j}$ are identically distributed, which implies that $\{\sum_{i=1}^{p_n+1-j} E_{i,j} / \sum_{i=1}^{n+1-j} E_{i,j},\ 1 \le j \le p_n\}$ and $\{Y_{nj}^2,\ 1 \le j \le p_n\}$ are identically distributed. For simplicity, we assume

$$Y_{nj}^2 = \frac{\sum_{i=1}^{p_n+1-j} E_{i,j}}{\sum_{i=1}^{n+1-j} E_{i,j}} = 1 - \frac{S_j}{T_{n+1-j}}, \text{ for } 1 \le j \le p_n \tag{3.1}$$

where $S_j = \sum_{i=p_n-j+2}^{n+1-j} E_{i,j}$ and $T_{n+1-j} = \sum_{i=1}^{n+1-j} E_{i,j}$. Then we have

$$P\Big(\max_{1\le j\le p_n} |\mathbf{z}_j|^2 \le t\Big) = P\Big(\max_{1\le j\le p_n} Y_{nj}^2 \le t\Big) = \prod_{j=1}^{p_n} F_{p_n+1-j:n-j}(t) \tag{3.2}$$

for $0 < t < 1$, and

$$1 - F_{1:k_n}(x) \le 1 - F_{2:k_n+1}(x) \le \cdots \le 1 - F_{p_n:n-1}(x)$$

for $x \in (0,1)$. See the proof of Theorem 2 in Jiang and Qi [17].

It is easily seen that $\{S_j,\ 1 \le j \le p_n\}$ are i.i.d. random variables with Gamma (k_n) distribution.

We will present some useful lemmas before we prove our main result.

Lemma 3.1 *(Gui and Qi [13]) Suppose* $\{l_n,\, n \ge 1\}$ *is a sequence of positive integers. Let* $z_{nj} \in [0,1)$ *be real numbers for* $1 \le j \le l_n$ *such that* $\max_{1\le j\le l_n} z_{nj} \to 0$ *as* $n \to \infty$*. Then* $\lim_{n\to\infty} \prod_{j=1}^{l_n} (1 - z_{nj}) \in (0,1)$ *exists if and only if the limit* $\lim_{n\to\infty} \sum_{j=1}^{l_n} z_{nj} =: z \in (0,\infty)$ *exists and the relationship of the two limits is given by*

$$\lim_{n\to\infty} \prod_{i=1}^{l_n} (1 - z_{ni}) = e^{-z}.$$

Lemma 3.2 *(Gui and Qi [13]) Assume that* $1 \le p_n < n$ *and* $p_n \to \infty$ *as* $n \to \infty$*. Let* $\{r_n\}$ *be a sequences of integers such that* $r_n < p_n$ *and* $p_n/r_n \to 1$ *as* $n \to \infty$*. Assume*

$\alpha_n > 0$ and β_n are real numbers such that $\lim_{n\to\infty} P(Y_{n1}^2 > \beta_n + \alpha_n x) = 0$ for any $x \in \mathbb{R}$. If $(\max_{1\le j\le r_n} Y_{nj}^2 - \beta_n)/\alpha_n$ converges in distribution to a cdf G, then $(\max_{1\le j\le p_n} Y_{nj}^2 - \beta_n)/\alpha_n$ converges in distribution to the same distribution G.

Lemma 3.3 *(Gui and Qi [13]) Let Z_n be nonnegative random variables such that $(Z_n^2 - \beta_n)/\alpha_n$ converges weakly to a cdf $G(x)$, where $\alpha_n > 0$ and $\beta_n > 0$ are constants satisfying that $\lim_{n\to\infty} \alpha_n/\beta_n = 0$. Then*

$$\frac{Z_n - \beta_n^{1/2}}{\alpha_n/(2\beta_n^{1/2})} \text{ converges weakly to } G.$$

Recall we just define that $\{S_j,\ 1 \le k \le p_n\}$ are i.i.d. Gamma(k_n) random variables. For convenience, we assume that $\{S_j,\ j \ge 1\}$ are i.i.d. random variables with Gamma(k_n) distribution.

Define the cumulative distribution function (cdf) of Gamma(a) random variable (incomplete gamma function) as

$$P(a, z) = \frac{1}{\Gamma(a)} \int_0^z x^{a-1} e^{-x} dx, \quad z \ge 0,$$

and the error function

$$\operatorname{erfc}(z) = \frac{2}{\sqrt{\pi}} \int_z^\infty e^{-x^2} dx, \quad z \ge 0.$$

Write

$$\tau(\lambda) = \lambda - 1 - \log \lambda, \quad \lambda > 0.$$

It is easy to see that for $\lambda > 0$

$$\tau(\lambda) = \lambda - 1 - \log \lambda = (1-\lambda)^2 \int_0^1 \frac{s}{1-(1-\lambda)s} ds.$$

We see that $\tau(\lambda) \ge 0$ for $\lambda > 0$. Since $\min(\lambda, 1) \le 1 - (1-\lambda)s \le \max(1, \lambda)$ for $0 < s < 1$, we have

$$\frac{1}{2}\frac{(1-\lambda)^2}{\max(1,\lambda)} \le \tau(\lambda) \le \frac{1}{2}\frac{(1-\lambda)^2}{\min(1,\lambda)} \quad \lambda > 0 \tag{3.3}$$

and conclude that

$$\sqrt{\tau(\lambda)} \ge \frac{1-\lambda}{\sqrt{2}}, \quad 0 < \lambda < 1. \tag{3.4}$$

We can also verify that for $s, t > 0$

$$\tau(st) = \tau(s) + \tau(t) + (s-1)(t-1). \tag{3.5}$$

This property will be used later.

Define

$$\phi(a, \lambda) = \frac{1}{\sqrt{2\pi a}} e^{-a\tau(\lambda)} = \frac{1}{\sqrt{2\pi a}} e^{-a(\lambda - 1 - \log \lambda)}, \quad \lambda > 0, \ a > 0. \tag{3.6}$$

Lemma 3.4 *Let δ_n be a sequence of positive numbers such that $\delta_n \to \infty$ and $\delta_n/\sqrt{k_n} \to 0$ as $n \to \infty$. Then*

$$P(k_n, k_n\lambda) = (1 + o(1)) \frac{1}{\sqrt{2\pi k_n}(1-\lambda)} \exp(-k_n \tau(\lambda)) \tag{3.7}$$

uniformly over $0 < \lambda \le 1 - \delta_n/\sqrt{k_n}$ as $n \to \infty$.

Proof. It follows from equations (2.15) and (4.3) in Temme [28] that

$$|P(a, a\lambda) - \frac{1}{2}\mathrm{erfc}(-\eta\sqrt{\frac{a}{2}}) + \frac{c_0(\lambda)}{\sqrt{2\pi a}} \exp(-\frac{1}{2}a\eta^2)| \le \frac{C}{a\sqrt{2\pi a}} \exp(-\frac{1}{2}a\eta^2) + \frac{Ce^a a^{-a}\Gamma(a)}{a\sqrt{2\pi a}} P(a, a\lambda) \tag{3.8}$$

holds uniformly for $0 < \lambda < 1$ and $a > 0$, where $C > 0$ is a universal constant, $c_0(\lambda) = \frac{1}{\lambda - 1} - \frac{1}{\eta}$, and

$$\eta = -(2(\lambda - 1 - \log\lambda))^{1/2} = -\sqrt{2\tau(\lambda)} \quad \text{for } 0 < \lambda < 1. \tag{3.9}$$

From Stirling's formula, see, e.g., Formula 6.1.38 in Abramowitz and Stegun [1]

$$\Gamma(a+1) = \sqrt{2\pi} a^{a+\frac{1}{2}} \exp(-a + \frac{\theta}{12a}), \quad \theta \in (0, 1)$$

we have

$$e^a a^{-a}\Gamma(a) = e^a a^{-a-1}\Gamma(a+1) \le \sqrt{2\pi} e^{1/12} a^{-1/2}$$

for all $a \ge 1$, which together with (3.8) implies that for some universal constant $C > 0$

$$|P(a, a\lambda) - \frac{1}{2}\mathrm{erfc}(-\eta\sqrt{\frac{a}{2}}) + \frac{c_0(\lambda)}{\sqrt{2\pi a}} \exp(-\frac{1}{2}a\eta^2)| \le \frac{C}{a\sqrt{2\pi a}} \exp(-\frac{1}{2}a\eta^2) + \frac{C}{a^2} P(a, a\lambda)$$

holds uniformly for $0 < \lambda < 1$ and $a > 1$. By setting

$$c(a, \lambda) = \frac{P(a, a\lambda) - \frac{1}{2}\mathrm{erfc}(-\eta\sqrt{\frac{a}{2}}) + \frac{c_0(\lambda)}{\sqrt{2\pi a}} \exp(-\frac{1}{2}a\eta^2)}{\frac{1}{a\sqrt{2\pi a}} \exp(-\frac{1}{2}a\eta^2) + \frac{1}{a^2} P(a, a\lambda)},$$

we have $|c(a,\lambda)| \le C$ for $0 < \lambda < 1$ and $a > 1$, and

$$P(a,a\lambda) - \frac{1}{2}\mathrm{erfc}(-\eta\sqrt{\frac{a}{2}}) + \frac{c_0(\lambda)}{\sqrt{2\pi a}}\exp(-\frac{1}{2}a\eta^2) = c(a,\lambda)(\frac{1}{a\sqrt{2\pi a}}\exp(-\frac{1}{2}a\eta^2) + \frac{1}{a^2}P(a,a\lambda)). \tag{3.10}$$

Assume $x > 0$. Since

$$\int_x^\infty \frac{e^{-t^2}}{t^2}dt < \frac{1}{x^3}\int_x^\infty te^{-t^2}dt = \frac{1}{2x^3}\int_x^\infty e^{-t^2}dt^2 = \frac{e^{-x^2}}{2x^3}$$

and by using integration by parts

$$\begin{aligned}\int_x^\infty e^{-t^2}dt &= -\frac{1}{2}\int_x^\infty \frac{de^{-t^2}}{t} \\ &= \frac{1}{2}\frac{e^{-x^2}}{x} - \frac{1}{2}\int_x^\infty \frac{e^{-t^2}}{t^2}dt,\end{aligned}$$

we get

$$(\frac{1}{2x} - \frac{1}{4x^3})e^{-x^2} < \int_x^\infty e^{-t^2}dt < \frac{1}{2x}e^{-x^2}$$

Therefore, we can define function $h(x)$ such that

$$\frac{1}{\sqrt{\pi}}\int_x^\infty e^{-t^2}dt = (\frac{1}{\sqrt{2}x} - \frac{h(x)}{\sqrt{2}x})\frac{1}{\sqrt{2\pi}}e^{-x^2}, \tag{3.11}$$

where $0 < h(x) < \frac{1}{2x^2}$ for $x > 0$.

By using (3.11) and (3.9), we have

$$\begin{aligned}&\frac{1}{2}\mathrm{erfc}(-\eta\sqrt{\frac{a}{2}}) - \frac{c_0(\lambda)}{\sqrt{2\pi a}}\exp(-\frac{1}{2}a\eta^2) \\ =\ & \frac{1}{2}\mathrm{erfc}(\sqrt{a\tau(\lambda)}) - (\frac{1}{\sqrt{2\tau(\lambda)}} - \frac{1}{1-\lambda})\frac{1}{\sqrt{2\pi a}}\exp(-a\tau(\lambda)) \\ =\ & \frac{1}{1-\lambda}\Big(1 - \frac{1-\lambda}{\sqrt{2\tau(\lambda)}}h(\sqrt{a\tau(\lambda)})\Big)\frac{1}{\sqrt{2\pi a}}\exp(-a\tau(\lambda)) \\ =\ & \Big(1 - \frac{1-\lambda}{\sqrt{2\tau(\lambda)}}h(\sqrt{a\tau(\lambda)})\Big)\frac{\phi(a,\lambda)}{1-\lambda}.\end{aligned}$$

Then it follows from (3.10) that

$$(1 - \frac{c(a,\lambda)}{a^2})P(a,a\lambda) = \Big(1 - \frac{1-\lambda}{\sqrt{2\tau(\lambda)}}h(\sqrt{a\tau(\lambda)}) + \frac{c(a,\lambda)(1-\lambda)}{a}\Big)\frac{\phi(a,\lambda)}{1-\lambda}$$

and thus

$$P(a,a\lambda) = \frac{\Big(1 - \frac{1-\lambda}{\sqrt{2\tau(\lambda)}}h(\sqrt{a\tau}) + \frac{c(a,\lambda)(1-\lambda)}{a}\Big)}{1 - \frac{c(a,\lambda)}{a^2}}\frac{\phi(a,\lambda)}{1-\lambda}$$

uniformly over $0 < \lambda < 1$ and $a > 1$.

Now let $a = k_n$. Let δ_n be any sequence of positive numbers such that $\delta_n \to \infty$ and $\delta_n/\sqrt{k_n} \to 0$ as $n \to \infty$. From (3.4) we have

$$\sqrt{k_n \tau(\lambda)} \geq \sqrt{\frac{k_n}{2}}(1-\lambda) \geq \frac{\delta_n}{\sqrt{2}} \to \infty$$

if $0 < \lambda \leq 1 - \delta_n/\sqrt{k_n}$, which implies $h(\sqrt{k_n\tau(\lambda)}) \to 0$ uniformly over $0 < \lambda \leq 1 - \delta_n/\sqrt{k_n}$ as $n \to \infty$. Therefore, we conclude that

$$P(k_n, k_n\lambda) = (1+o(1))\frac{\phi(a,\lambda)}{1-\lambda} = (1+o(1))\frac{1}{\sqrt{2\pi k_n}(1-\lambda)}\exp(-k_n\tau(\lambda))$$

uniformly over $0 < \lambda \leq 1 - \delta_n/\sqrt{k_n}$ as $n \to \infty$, i.e. (3.7) holds. This completes the proof of the lemma. ■

Lemma 3.5 *Under condition* (2.7) *we have*

$$\sqrt{k_n}(1-\lambda_n) \to \infty \qquad \textit{as } n \to \infty \tag{3.12}$$

and

$$\sqrt{k_n}(1-\lambda_n) = O(\sqrt{\log n}) \qquad \textit{as } n \to \infty. \tag{3.13}$$

Proof. Since

$$g_n(\lambda) = \lambda - 1 - \log(\lambda) + \frac{2}{k_n}\log(1-\lambda)$$

is decreasing in $\lambda \in (0,1)$, we have for any $\delta > 0$, $\sqrt{k_n}(1-\lambda_n) > \delta$ if and only if $g_n(1 - \delta/\sqrt{k_n}) < \frac{1}{k_n}\log(\frac{n}{2\pi k_n^{3/2}})$. To prove (3.12), it suffice to show that for any $\delta > 0$, $g_n(1 - \delta/\sqrt{k_n}) < \frac{1}{k_n}\log(\frac{n}{2\pi k_n^{3/2}})$ for all large n. In fact, for any fixed $\delta > 0$, we have from (3.3) that for all large n

$$\begin{aligned} g_n(1-\frac{\delta}{\sqrt{k_n}}) &= \tau(1-\frac{\delta}{\sqrt{k_n}}) + \frac{2}{k_n}\log(\frac{\delta}{\sqrt{k_n}}) \\ &\leq \frac{1}{2}\frac{\delta^2}{k_n(1-\delta/\sqrt{k_n})} + \frac{2}{k_n}\log(\frac{\delta}{\sqrt{k_n}}) \\ &\leq \frac{\delta^2}{k_n} + \frac{2}{k_n}\log(\frac{\delta}{\sqrt{k_n}}) \\ &< \frac{1}{k_n}\log(\frac{n}{2\pi\delta^2\sqrt{k_n}}) + \frac{1}{k_n}\log(\frac{\delta^2}{k_n}) \\ &= \frac{1}{k_n}\log(\frac{n}{2\pi k_n^{3/2}}), \end{aligned}$$

proving (3.12).

Now we prove (3.13). By using (3.4), we have

$$(1-\lambda_n)^2 \le 2\tau(\lambda_n) = 2g(\lambda_n) - \frac{4}{k_n}\log(1-\lambda_n) = \frac{2}{k_n}\log(\frac{n\sqrt{k_n}}{2\pi}) - \frac{4}{k_n}\log(k_n(1-\lambda_n)).$$

From (3.12), we have $\log(k_n(1-\lambda_n)) > 0$ for all large n. Therefore, we get

$$k_n(1-\lambda_n)^2 < 2\log(\frac{n\sqrt{k_n}}{2\pi}) = O(\log n) \quad \text{as } n\to\infty,$$

which proves (3.13). ■

For convenience, we will introduce more notations for the rest of the paper.

Define for $x\in\mathbb{R}$

$$\lambda_n(x) = \lambda_n(1+\frac{x}{k_n(1-\lambda_n)}), \tag{3.14}$$

and for $1\le j<n$

$$\lambda_{n,j}(x) = \frac{n+1-j}{n}\lambda_n(x). \tag{3.15}$$

Lemma 3.6 *Assume condition* (2.7) *holds. We have for any fixed* $x\in\mathbb{R}$ *that*

$$\frac{1-\lambda_n(x)}{1-\lambda_n} - 1 \to 0. \tag{3.16}$$

If further we assume that $\{j_n\}$ *is a sequence of positive integers with* $1<j_n<n-1$ *such that*

$$\frac{k_n j_n(1-\lambda_n)}{n} \to \infty \quad \textit{and} \quad \frac{k_n j_n^2}{n^2} \to 0, \tag{3.17}$$

then for any fixed $x\in\mathbb{R}$

$$\max_{1\le j\le j_n} |\frac{1-\lambda_{n,j}(x)}{1-\lambda_n} - 1| \to 0. \tag{3.18}$$

Proof. The proofs are omitted here since they are straightforward by using Lemma 3.5 and given conditions. ■

Lemma 3.7 *Assume condition* (2.7) *holds. Then with* $\phi(a,\lambda)$ *defined in* (3.6) *we have for* $x\in\mathbb{R}$

$$\frac{1}{(1-\lambda_n)^2}\phi(k_n,\lambda_n(x)) = (1+o(1))\frac{k_n e^x}{n} \quad \textit{as } n\to\infty. \tag{3.19}$$

Proof. Fix $x \in \mathbb{R}$. It follows from (3.12) and (3.16) that

$$\delta_n(x) := \sqrt{k_n}(1-\lambda_n(x)) = \sqrt{k_n}(1-\lambda_n)(1+o(1)) \to \infty.$$

Then $0 < \lambda_n(x) < 1$ for all large n, which will be used in the proof of Lemma 3.8. By using (3.5) we have from (3.3)

$$\begin{aligned}
\tau(\lambda_n(x)) &= \tau(\lambda_n) + \tau(1+\frac{x}{k_n(1-\lambda_n)}) + (\lambda_n - 1)\frac{x}{k_n(1-\lambda_n)} \\
&= \tau(\lambda_n) + \tau(1+\frac{x}{k_n(1-\lambda_n)}) - \frac{x}{k_n} \\
&= g_n(\lambda_n) - \frac{2}{k_n}\log(1-\lambda_n) + \frac{x}{k_n} + \tau(1+\frac{x}{k_n(1-\lambda_n)}) \\
&= \frac{1}{k_n}\log(\frac{n}{\sqrt{2\pi}k_n^{3/2}}) - \frac{2}{k_n}\log(1-\lambda_n) - \frac{x}{k_n} + O(\frac{1}{k_n^2(1-\lambda_n)^2}).
\end{aligned}$$

Therefore, we obtain

$$\begin{aligned}
& \frac{1}{(1-\lambda_n)^2}\phi(k_n, \lambda_n(x)) \\
= & \frac{1}{\sqrt{2\pi k_n}(1-\lambda_n)^2} e^{-k_n \tau(\lambda_n(x))} \\
= & \frac{1}{\sqrt{2\pi k_n}(1-\lambda_n)^2} \exp\Big\{ -\log(\frac{n}{\sqrt{2\pi}k_n^{3/2}}) + 2\log(1-\lambda_n) + x + O(\frac{1}{k_n(1-\lambda_n)^2})\Big\} \\
= & \frac{k_n e^x}{n} \exp\Big\{O(\frac{1}{k_n(1-\lambda_n)^2})\Big\} \\
= & \frac{k_n e^x}{n}(1+o(1)).
\end{aligned}$$

This completes the proof. ■

Lemma 3.8 *Fix* $x \in \mathbb{R}$. *Assume* $\{j_n\}$ *is a sequence of positive integers with* $1 < j_n < n-1$ *such that* (3.17) *holds. Then*

$$\sum_{j=1}^{j_n} P(k_n, \frac{n+1-j}{n}\lambda_n(x)) \to e^x. \tag{3.20}$$

Furthermore, if $\{q_n\}$ *is a sequence of positive integers such that* $1 < j_n \le q_n < n-1$ *, then*

$$\sum_{j=1}^{q_n} P(k_n, \frac{n+1-j}{n}\lambda_n(x)) \to e^x. \tag{3.21}$$

Proof. It follows from (3.12) and (3.16) that $\delta_n(x) = \sqrt{k_n}(1-\lambda_n(x)) \to \infty$. Since $0 < \lambda_{n,j} \le \lambda_n(x) \le 1 - \delta_n/\sqrt{k_n}$ for all $1 \le j < n$, we have from (3.7) that

$$P(k_n, k_n\lambda_{n,j}(x)) = (1+o(1))\frac{1}{\sqrt{2\pi k_n}(1-\lambda_{n,j}(x))}\exp(-k_n\tau(\lambda_{n,j}(x))) \tag{3.22}$$

uniformly over $1 \le j < n$ as $n \to \infty$.

Since $\lambda_{n,j}(x) = \frac{n+1-j}{n}\lambda_n(x)$, we have from (3.5) that

$$\tau(\lambda_{n,j}(x)) = \tau(\lambda_n(x)) + \tau(1-\frac{j-1}{n}) + (1-\lambda_n(x))\frac{j-1}{n}.$$

From (3.22) we have

$$P(k_n, k_n\lambda_{n,j}(x)) = (1+o(1))\frac{\phi(k_n,\lambda_n(x))}{1-\lambda_{n,j}(x)}\exp\{-k_n\tau(1-\frac{j-1}{n}) - \frac{(j-1)(1-\lambda_n(x))k_n}{n}\} \tag{3.23}$$

uniformly over $1 \le j < n$ as $n \to \infty$.

Note that (3.18) and (3.16) hold under (3.17). Then we have from (3.23) and (3.3) that

$$\begin{aligned}
P(k_n, k_n\lambda_{n,j}(x)) &= (1+o(1))\frac{\phi(k_n,\lambda_n(x))}{1-\lambda_n(x)}\exp\{-k_nO(\frac{j_n^2}{n^2}) - \frac{(j-1)(1-\lambda_n(x))k_n}{n}\} \\
&= (1+o(1))\frac{\phi(k_n,\lambda_n(x))}{1-\lambda_n(x)}\exp\{o(1) - \frac{(j-1)(1-\lambda_n(x))k_n}{n}\} \\
&= (1+o(1))\frac{\phi(k_n,\lambda_n(x))}{1-\lambda_n(x)}\exp\{-\frac{(j-1)(1-\lambda_n(x))k_n}{n}\}
\end{aligned}$$

uniformly over $1 \le j < j_n$ as $n \to \infty$, which yields that

$$\begin{aligned}
\sum_{j=1}^{j_n} P(k_n, k_n\lambda_{n,j}(x)) &= (1+o(1))\frac{\phi(k_n,\lambda_n(x))}{1-\lambda_n(x)}\sum_{j=1}^{j_n}\Big(\exp\{-\frac{(1-\lambda_n(x))k_n}{n}\}\Big)^{j-1} \\
&= (1+o(1))\frac{\phi(k_n,\lambda_n(x))}{1-\lambda_n(x)}\frac{1-\exp\{-\frac{(1-\lambda_n(x))(j_n+1)k_n}{n}\}}{1-\exp\{-\frac{(1-\lambda_n(x))k_n}{n}\}} \\
&= (1+o(1))\frac{\phi(k_n,\lambda_n(x))}{1-\lambda_n(x)}\frac{1-\exp\{-(1+o(1))\frac{(1-\lambda_n)(j_n+1)k_n}{n}\}}{\frac{(1-\lambda_n(x))k_n}{n}} \\
&= (1+o(1))\frac{\phi(k_n,\lambda_n(x))}{(1-\lambda_n(x))^2}\frac{n}{k_n} \\
&= (1+o(1))\frac{\phi(k_n,\lambda_n(x))}{(1-\lambda_n)^2}\frac{n}{k_n} \\
&= (1+o(1))e^x
\end{aligned}$$

from (3.19).

Next, we prove (3.21) when $q_n > j_n$. Note that $\tau(1-\frac{j-1}{n}) \geq 0$, and $1-\lambda_{n,j}(x) \geq 1-\lambda_n(x)$ for $1 \leq j \leq q_n$. Then from (3.23) we have

$$\begin{aligned}
\sum_{j=1}^{q_n} P(k_n, k_n\lambda_{n,j}(x)) &\leq (1+o(1))\frac{\phi(k_n, \lambda_n(x))}{1-\lambda_{n,j}(x)}\sum_{j=1}^{q_n}\exp\{- - \frac{(j-1)(1-\lambda_n(x))k_n}{n}\} \\
&= (1+o(1))\frac{\phi(k_n, \lambda_n(x))}{1-\lambda_{n,j}(x)}\sum_{j=1}^{q_n}\Big(\exp\{-\frac{(1-\lambda_n(x))k_n}{n}\}\Big)^{j-1} \\
&\leq (1+o(1))\frac{\phi(k_n, \lambda_n(x))}{1-\lambda_n(x)}\frac{1}{1-\exp\{-\frac{(1-\lambda_n(x))k_n}{n}\}} \\
&= (1+o(1))\frac{\phi(k_n, \lambda_n(x))}{1-\lambda_n(x)}\frac{1}{\frac{(1-\lambda_n(x))k_n}{n}} \\
&= (1+o(1))\frac{\phi(k_n, \lambda_n(x))}{(1-\lambda_n(x))^2}\frac{n}{k_n} \\
&= (1+o(1))\frac{\phi(k_n, \lambda_n(x))}{(1-\lambda_n)^2}\frac{n}{k_n} \\
&= (1+o(1))e^x
\end{aligned}$$

from (3.19), which together with (3.20), implies (3.21) since

$$\sum_{j=1}^{j_n} P(k_n, k_n\lambda_{n,j}(x)) \leq \sum_{j=1}^{q_n} P(k_n, k_n\lambda_{n,j}(x)).$$

This completes the proof of the lemma. ■

Lemma 3.9 *Assume $x_1, \cdots, x_n$ are positive numbers for $n \geq 1$ and $\varepsilon_{n,i}$, $1 \leq i \leq n$ are such that $\varepsilon_n = \max_{1\leq i\leq n}|\varepsilon_{n,i}| < 1$. Then*

$$|\min_{1\leq i\leq n} x_i(1+\varepsilon_{n,i}) - \min_{1\leq i\leq n} x_i| \leq \varepsilon_n \min_{1\leq i\leq n} x_i.$$

Proof. For $1 \leq i \leq n$ we have

$$x_i(1-\varepsilon_n) \leq x_i(1+\varepsilon_{n,i}) \leq x_i(1+\varepsilon_n),$$

and thus we obtain that

$$\min_{1\leq i\leq n} x_i(1-\varepsilon_n) \leq \min_{1\leq i\leq n} x_i(1+\varepsilon_{n,i}) \leq \min_{1\leq i\leq n} x_i(1+\varepsilon_n),$$

which implies

$$-\varepsilon_n \min_{1\le i\le n} x_i \le \min_{1\le i\le n} x_i(1+\varepsilon_{n,i}) - \min_{1\le i\le n} x_i \le \varepsilon_n \min_{1\le i\le n} x_i.$$

This completes the proof of the lemma. ■

Lemma 3.10 *Under condition* (2.7) *we have*

$$\lim_{n\to\infty} P(Y_{n1}^2 > 1 - \frac{k_n}{n}\lambda_n(-x)) = \lim_{n\to\infty} P(Y_{n1}^2 > 1 - \frac{k_n\lambda_n}{n} + \frac{\lambda_n x}{n(1-\lambda_n)}) = 0$$

for each $x \in \mathbb{R}$.

Proof. By using expression (3.1) we have $Y_{n1}^2 = 1 - \frac{S_1}{T_n}$, where S_1 has a Gamma (k_n) distribution and T_n has a Gamma(n) distribution. From the central limit theorem we have

$$V_{n1} := \frac{S_1 - k_n}{k_n^{1/2}} \xrightarrow{d} N(0,1), \quad V_{n2} := \frac{T_n - n}{n^{1/2}} \xrightarrow{d} N(0,1).$$

Then we have

$$Y_{n1}^2 = 1 - \frac{k_n + k_n^{1/2}V_{n1}}{n + n^{1/2}V_{n2}} = 1 - \frac{k_n}{n}\frac{1 + V_{n1}/k_n^{1/2}}{1 + V_{n2}/n^{1/2}} = 1 - \frac{k_n}{n}(1 + \frac{V_{n1}}{k_n^{1/2}} + O_p(n^{-1/2})),$$

and thus we get

$$V_{n3} := \frac{n}{k_n^{1/2}}(Y_{n1}^2 - 1 + \frac{k_n}{n}) = V_{n1} + O_p(\sqrt{\frac{k_n}{n}}) \xrightarrow{d} N(0,1),$$

which yields

$$P(Y_{n1}^2 > 1 - \frac{k_n\lambda_n}{n} + \frac{\lambda_n x}{n(1-\lambda_n)}) = P(V_{n3} > \sqrt{k_n}(1-\lambda_n) + \frac{\lambda_n x}{\sqrt{k_n}(1-\lambda_n)}) \to 0$$

since $\sqrt{k_n}(1-\lambda_n) \to \infty$ as $n \to \infty$ from (3.12). This completes of the proof. ■

Lemma 3.11 *Let* $\{T_j,\ j \ge 1\}$ *be a sequence of random variables, and for each* $j \ge 1$, T_j *has a Gamma (j) distribution with density function* $t^{j-1}e^{-t}I(t>0)/(j-1)!$. *Then*

$$\max_{m_n \le j \le n} |\frac{T_j}{j} - 1| = O(\frac{\sqrt{\log n}}{\sqrt{m_n}}) \quad \textit{almost surely (a.s.)},$$

where m_n *is any sequence of integers such that* $1 \le m_n < n$ *and* $m_n/(\log n)^3 \to \infty$ *as* $n \to \infty$.

Proof. Set $\tau_n = \frac{\sqrt{\log n}}{\sqrt{m_n}}$. By Theorem 1 on page 217 in Petrov [24],

$$P(T_j - j > x\sqrt{j}) = (1+o(1))(1-\Phi(x)), \quad P(T_j - j < -x\sqrt{j}) = (1+o(1))(1-\Phi(x))$$

uniformly over $0 \le x \le d_j$ as $j \to \infty$, where Φ is the cumulative distribution function for the standard normal random variable, d_j is any sequence of positive numbers with $d_j = o(j^{1/6})$. By setting $x = 4\sqrt{\log n}$ when $m_n \le j \le n$ and using the approximation $1-\Phi(x) \sim \frac{1}{\sqrt{2\pi}x}e^{-x^2/2} = \frac{1}{\sqrt{2\pi}4n^8\sqrt{\log n}}$ we conclude that

$$\begin{aligned}
& P(\max_{m_n \le j \le n} |\frac{T_j}{j} - 1| > 2\tau_n) \\
\le & \sum_{j=m_n}^{n} P(|\frac{T_j}{j} - 1| > 2\tau_n) \\
\le & \sum_{j=m_n}^{n} P(|T_j - j| > 2j\tau_n) \\
\le & \sum_{j=m_n}^{n} P(|T_j - j| > 2\sqrt{\log n}\sqrt{j}) \\
\le & \sum_{j=m_n}^{n} P(T_j - j > 2\sqrt{\log n}\sqrt{j}) + \sum_{j=m_n}^{n} P(T_j - j < -2\sqrt{\log n}\sqrt{j}) \\
\sim & \frac{2(n - m_n + 1)}{\sqrt{2\pi}4n^8\sqrt{\log n}},
\end{aligned}$$

which implies that $\sum_{n>5} P(\max_{m_n \le j \le n} |\frac{T_j}{j} - 1| > 2\tau_n) < \infty$. The lemma is proved by using the Borel-Cantelli lemma. ∎

From now on, we define $m_n = [k_n(\log n)^3]$, the integer part of $k_n(\log n)^3$. Then $m_n > k_n$ for all large n and $m_n/(\log n)^3 \to \infty$ as $n \to \infty$.

Lemma 3.12 *Under condition* (2.7) *we have*

$$L_n := (\frac{\lambda_n}{n(1-\lambda_n)})^{-1}(\min_{1 \le j \le n-m_n} \frac{S_j}{n+1-j} - \frac{k_n\lambda_n}{n}) \xrightarrow{d} \Lambda_1 \tag{3.24}$$

where $\Lambda_1(x) = 1 - \Lambda(-x)$, $x \in \mathbb{R}$.

Proof. Fix $x \in \mathbb{R}$. Recall $\lambda_n(x)$ and $\lambda_{n,j}(x)$ are defined in (3.14) and (3.15), respectively.

We have

$$
\begin{aligned}
& P\Big((\frac{\lambda_n}{n(1-\lambda_n)})^{-1} (\min_{1 \le j \le n-m_n} \frac{S_j}{n+1-j} - \frac{k_n \lambda_n}{n}) \le x \Big) \\
= \;& P\Big(\min_{1 \le j \le n-m_n} \frac{S_j}{n+1-j} \le \frac{k_n \lambda_n}{n} + \frac{\lambda_n}{n(1-\lambda_n)} x \Big) \\
= \;& P\Big(\min_{1 \le j \le n-m_n} \frac{S_j}{n+1-j} \le \frac{k_n}{n} \lambda_n(x) \Big) \\
= \;& 1 - P\Big(\min_{1 \le j \le n-m_n} \frac{S_j}{n+1-j} > \frac{k_n}{n} \lambda_n(x) \Big) \\
= \;& 1 - \prod_{j=1}^{n-m_n} P\Big(\frac{S_j}{n+1-j} > \frac{k_n}{n} \lambda_n(x) \Big) \\
= \;& 1 - \prod_{j=1}^{n-m_n} P\Big(S_j > k_n \lambda_{n,j}(x) \Big) \\
= \;& 1 - \prod_{j=1}^{n-m_n} (1 - z_{nj}),
\end{aligned}
$$

where $z_{nj} = P(k_n, k_n \lambda_{n,j}(x))$ for $1 \le j \le n - m_n$. Note that z_{nj} is non-increasing in j, and $\lambda_{n,1}(x) = \lambda_n(x)$. It follows from (3.23) with $j = 1$, (3.19) and (3.16) that

$$
z_{n1} = (1 + o(1)) \frac{\phi(k_n, \lambda_n(x))}{1 - \lambda_n(x)} \le 1 + o(1)) \frac{\phi(k_n, \lambda_n(x))}{(1 - \lambda_n(x))^2} = (1 + o(1)) \frac{k_n e^x}{n} \to 0
$$

as $n \to \infty$.

To apply Lemma 3.8, we define $\delta_n = k_n^{1/2}(1 - \lambda_n)$. Then $\delta_n \to \infty$ from (3.12). Set $j_n = [n/\sqrt{k_n \delta_n}]$, the integer part of $n/\sqrt{k_n \delta_n}$. Then as $n \to \infty$

$$
\frac{k_n j_n (1 - \lambda_n)}{n} = \frac{k_n^{1/2} j_n \delta_n}{n} \sim \sqrt{\delta_n} \to \infty
$$

and

$$
\frac{k_n j_n^2}{n^2} \sim \frac{1}{\delta_n} \to 0,
$$

i.e. (3.17) holds. Obviously we have $n - m_n > j_n$ for all large n. Therefore, by applying Lemma 3.8 with $q_n = n - m_n$, we have

$$
\sum_{j=1}^{n-m_n} z_{nj} \to e^x,
$$

which coupled with Lemma 3.1 yields that $\prod_{j=1}^{n-m_n} (1 - z_{nj}) \to \exp(-e^x) = \Lambda(-x)$ as $n \to \infty$. Hence, we get

$$
P\Big((\frac{\lambda_n}{n(1-\lambda_n)})^{-1} (\min_{1 \le j \le n-m_n} \frac{S_j}{n+1-j} - \frac{k_n \lambda_n}{n}) \le x \Big) \to 1 - \Lambda(-x) = \Lambda_1(x),
$$

which proves the lemma. ■

Lemma 3.13 *Under condition* (2.7) *we have*

$$(\frac{\lambda_n}{n(1-\lambda_n)})^{-1}(\max_{1\le j\le n-m_n} Y_{nj}^2 - (1-\frac{k_n\lambda_n}{n})) \xrightarrow{d} \Lambda.$$

Proof. From Lemma 3.11 we have $\varepsilon_n := \max_{m_n<j\le n}|\frac{T_j}{j}-1| = \max_{1\le j\le n-m_n}|\frac{T_{n+1-j}}{n+1-j}-1| = O(\frac{\sqrt{\log n}}{\sqrt{m_n}}) \to 0$ a.s. as $n\to\infty$. Then we have

$$1-\varepsilon_n \le \frac{T_{n+1-j}}{n+1-j} \le 1+\varepsilon_n \quad \text{uniformly for } 1\le j\le n-m_m$$

for all large n, i.e.

$$1-\frac{\varepsilon_n}{1-\varepsilon_n} \le \frac{1}{1+\varepsilon_n} \le \frac{n+1-j}{T_{n+1-j}} \le \frac{1}{1-\varepsilon_n} = 1+\frac{\varepsilon_n}{1-\varepsilon_n} \quad \text{uniformly for } 1\le j\le n-m_m$$

for large n. By writing $\varepsilon_{nj} = \frac{n+1-j}{T_{n+1-j}}-1$, we have from (3.1) that $Y_{nj}^2 = 1-\frac{S_j}{n+1-j}(1+\varepsilon_{nj})$, and thus

$$\max_{1\le j\le n-m_n} Y_{nj}^2 = 1-\min_{1\le j\le n-m_n}\frac{S_j}{n+1-j}(1+\varepsilon_{nj}).$$

Recall L_n is defined in equation (3.24). The above equation, together with Lemma 3.9, yields that for all large n

$$\begin{aligned}\Delta_n : &= |(\frac{\lambda_n}{n(1-\lambda_n)})^{-1}\Big(\max_{1\le j\le n-m_n} Y_{nj}^2-(1-\frac{k_n\lambda_n}{n})\Big)+L_n| \qquad (3.25)\\ &= (\frac{\lambda_n}{n(1-\lambda_n)})^{-1}|\min_{1\le j\le n-m_n}\frac{S_j}{n+1-j}(1+\varepsilon_{nj})-\min_{1\le j\le n-m_n}\frac{S_j}{n+1-j}|\\ &\le \frac{\varepsilon_n}{1-\varepsilon_n}(\frac{\lambda_n}{n(1-\lambda_n)})^{-1}\min_{1\le j\le n-m_n}\frac{S_j}{n+1-j}.\end{aligned}$$

Now we have from definition of L_n in equation (3.24) that

$$\min_{1\le j\le n-m_n}\frac{S_j}{n+1-j} = \frac{\lambda_n}{n(1-\lambda_n)}L_n+\frac{k_n\lambda_n}{n},$$

and thus obtain

$$\begin{aligned}\Delta_n &\le \frac{\varepsilon_n}{1-\varepsilon_n}(\frac{\lambda_n}{n(1-\lambda_n)})^{-1}(\frac{\lambda_n}{n(1-\lambda_n)}L_n+\frac{k_n\lambda_n}{n})\\ &= \frac{\varepsilon_n L_n}{1-\varepsilon_n}+\frac{\varepsilon_n}{1-\varepsilon_n}k_n(1-\lambda_n)\\ &= o_p(1)+O_p(\frac{\sqrt{k_n}(1-\lambda_n)}{\log n})\\ &= o_p(1)\end{aligned}$$

from (3.13). Then it follows from (3.25) that $(\frac{\lambda_n}{n(1-\lambda_n)})^{-1}\Big(\max_{1\le j\le n-m_n} Y_{nj}^2 - (1-\frac{k_n\lambda_n}{n})\Big)$ and $-L_n$ have the same asymptotic distribution. The lemma follows since $-L_n \stackrel{d}{\to} \Lambda$ from Lemma 3.12. ■

Proof of Theorem 4. Set $\beta_n = 1 - \frac{k_n\lambda_n}{n}$ and $\alpha_n = \frac{\lambda_n}{n(1-\lambda_n)}$. We apply Lemma 3.2 with $r_n = n - m_n$ and $p_n = n - k_n$ under condition (2.7). Note that $\lim_{n\to\infty} P(Y_{n1}^2 > \beta_n + \alpha_n x) = 0$ from Lemma 3.10. Then from Lemmas 3.10 and 3.13 we have

$$\frac{\max_{1\le j\le p_n} Y_{nj}^2 - \beta_n}{\alpha_n} \stackrel{d}{\to} \Lambda.$$

Then from Lemma 3.3 we get

$$\frac{\max_{1\le j\le n-m_n} Y_{nj} - \beta_n^{1/2}}{\alpha_n/(2\beta_n^{1/2})} \stackrel{d}{\to} \Lambda,$$

i.e. Theorem 4 holds in view of (3.2). This completes the proof. ■

Acknowledgements. The authors would like to thank the referee for his/her constructive suggestions for revision. The research of Yu Miao was supported in part by NSFC (11971154). The research of Yongcheng Qi was supported in part by NSF Grant DMS-1916014.

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