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A CHARACTERIZATION OF A NEW TYPE OF STRONG LAW OF LARGE NUMBERS

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Abstract. Let $0 < p < 2$ and $1 \le q < \infty$. Let $\{X_n;\ n \ge 1\}$ be a sequence of independent copies of a real-valued random variable X and set $S_n = X_1 + \cdots + X_n,\ n \ge 1$. We say X satisfies the (p,q)*-type strong law of large numbers* (and write $X \in SLLN(p,q)$) if $\sum_{n=1}^{\infty} \frac{1}{n}\left(\frac{|S_n|}{n^{1/p}}\right)^q < \infty$ almost surely. This paper is devoted to a characterization of $X \in SLLN(p,q)$. By applying results obtained from the new versions of the classical Lévy, Ottaviani, and Hoffmann-Jørgensen (1974) inequalities proved by Li and Rosalsky (2013) and by using techniques developed by Hechner (2009) and Hechner and Heinkel (2010), we obtain sets of necessary and sufficient conditions for $X \in SLLN(p,q)$ for the six cases: $1 \le q < p < 2$, $1 < p = q < 2$, $1 < p < 2$ and $q > p$, $q = p = 1$, $p = 1 < q$, and $0 < p < 1 \le q$. The necessary and sufficient conditions for $X \in SLLN(p,1)$ have been discovered by Li, Qi, and Rosalsky (2011). Versions of above results in a Banach space setting are also given. Illustrative examples are presented.

1. Introduction

Throughout, let $(\mathbf{B}, \|\cdot\|)$ be a real separable Banach space equipped with its Borel σ-algebra $\mathcal{B}$ (= the σ-algebra generated by the class of open subsets of $\mathbf{B}$ determined by $\|\cdot\|$) and let $\{X_n;\ n \ge 1\}$ be a sequence of independent copies of a $\mathbf{B}$-valued random variable X defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$. As usual, let $S_n = \sum_{k=1}^{n} X_k,\ n \ge 1$ denote their partial sums. If $0 < p < 2$ and if X is a real-valued random variable (that is, if $\mathbf{B} = \mathbb{R}$), then

$$\lim_{n\to\infty} \frac{S_n}{n^{1/p}} = 0 \text{ almost surely (a.s.)}$$

if and only if $\mathbb{E}|X|^p < \infty$ where $\mathbb{E}X = 0$ whenever $p \ge 1$. This is the celebrated Kolmogorov-Marcinkiewicz-Zygmund strong law of large numbers (SLLN); see Kolmogoroff [8] for $p = 1$ and Marcinkiewicz and Zygmund [13] for $p \ne 1$.

The classical Kolmogorov SLLN in real separable Banach spaces was established by Mourier [16]. The extension of the Kolmogorov-Marcinkiewicz-Zygmund SLLN to $\mathbf{B}$-valued random variables is independently due to Azlarov and Volodin [1, Theorem] and de Acosta [3, Theorem 3.1]. De Acosta [3, Theorem 4.1] also provided

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a remarkable characterization of Rademacher type p $(1 \leq p < 2)$ Banach spaces via the SLLN. We refer to Ledoux and Talagrand [9] for the definitions of Rademacher type p and stable type p Banach spaces.

At the origin of the current investigation is the following recent and striking result by Hechner [4, Theorem 2.4.1] for $p = 1$ and Hechner and Heinkel [5, Theorem 5] for $1 < p < 2$ which are new even in the case where the Banach space $\mathbf{B}$ is the real line.

Theorem 1.1. (Hechner [4, Theorem 2.4.1] for $p = 1$ and Hechner and Heinkel [5, Theorem 5] for $1 < p < 2$). *Suppose that* $\mathbf{B}$ *is of stable type* p *for some* $p \in [1, 2)$ *and let* $\{X_n;\ n \geq 1\}$ *be a sequence of independent copies of a* $\mathbf{B}$*-valued variable* X *with* $\mathbb{E}X = 0$. *Then*

$$\sum_{n=1}^{\infty} \frac{1}{n} \left(\frac{\mathbb{E}\|S_n\|}{n^{1/p}} \right) < \infty$$

if and only if

$$\begin{cases} \mathbb{E}\|X\| \ln(1 + \|X\|) < \infty & \text{if } p = 1, \\ \displaystyle\int_0^{\infty} \mathbb{P}^{1/p}\left(\|X\| > t\right) dt < \infty & \text{if } 1 < p < 2. \end{cases}$$

Inspired by the above discovery by Hechner [4] and Hechner and Heinkel [5], Li, Qi, and Rosalsky [10] obtained sets of necessary and sufficient conditions for

$$\sum_{n=1}^{\infty} \frac{1}{n} \left(\frac{\|S_n\|}{n^{1/p}} \right) < \infty \text{ a.s.}$$

for the three cases: $0 < p < 1$, $p = 1$, $1 < p < 2$ (see Theorem 2.4, Theorem 2.3, and Corollary 2.1, respectively of Li, Qi, and Rosalsky [10]). Again, these results are new when $\mathbf{B} = \mathbb{R}$; see Theorem 2.5 of Li, Qi, and Rosalsky [10]. Moreover for $1 \leq p < 2$, Li, Qi, and Rosalsky [10, Theorems 2.1 and 2.2] obtained necessary and sufficient conditions for

$$\sum_{n=1}^{\infty} \frac{1}{n} \left(\frac{\mathbb{E}\|S_n\|}{n^{1/p}} \right) < \infty$$

for general separable Banach spaces.

Motivated by the results obtained by Li, Qi, and Rosalsky [10], we introduce a new type strong law of large numbers as follows.

Definition 1.1. Let $0 < p < 2$ and $0 < q < \infty$. Let $\{X_n;\ n \geq 1\}$ be a sequence of independent copies of a $\mathbf{B}$-valued random variable X. We say X satisfies the (p, q)-type strong law of large numbers (and write $X \in SLLN(p, q)$) if

$$\sum_{n=1}^{\infty} \frac{1}{n} \left(\frac{\|S_n\|}{n^{1/p}} \right)^q < \infty \text{ a.s.}$$

The following result was recently obtained by Li, Qi, and Rosalsky [11, Theorem 1.3] who proved it by employing new versions of the classical Lévy, Ottaviani, and Hoffmann-Jørgensen [6] inequalities established by Li and Rosalsky [12] and by using some of techniques developed by Hechner and Heinkel [5]. Note that no conditions are imposed on the Banach space $\mathbf{B}$. Theorem 1.2 will be used in the proofs of the main results of the current work.

Theorem 1.2. *Let $0 < p < 2$ and $0 < q < \infty$. Let $\{X_n;\ n \geq 1\}$ be a sequence of independent copies of a $\mathbf{B}$-valued random variable X. Then*

$$\sum_{n=1}^{\infty} \frac{1}{n} \mathbb{E}\left(\frac{\|S_n\|}{n^{1/p}}\right)^q < \infty \tag{1.1}$$

if and only if

$$X \in SLLN(p,q) \tag{1.2}$$

and

$$\begin{cases} \displaystyle\int_0^{\infty} \mathbb{P}^{q/p}\left(\|X\|^q > t\right) dt < \infty & \text{if } 0 < q < p, \\ \mathbb{E}\|X\|^p \ln(1 + \|X\|) < \infty & \text{if } q = p, \\ \mathbb{E}\|X\|^q < \infty & \text{if } q > p. \end{cases} \tag{1.3}$$

Furthermore, each of (1.1) and (1.2) implies that

$$\lim_{n \to \infty} \frac{S_n}{n^{1/p}} = 0 \text{ a.s.} \tag{1.4}$$

For $0 < q < p$, (1.1) and (1.2) are equivalent so that each of them implies that (1.3) and (1.4) hold.

Remark 1.1. Let $q = 1$. Then one can easily see that Theorems 2.1 and 2.2 of Li, Qi, and Rosalsky [10] follow from Theorem 1.2.

Remark 1.2. It follows from the conclusion (1.4) of Theorem 1.2 that, if (1.2) holds for some $q = q_1 > 0$ then (1.2) holds for all $q > q_1$.

The current work continues the investigations by Hechner [4], Hechner and Heinkel [5], and Li, Qi, and Rosalsky [10] and [11]. More specifically:

- **(i)** For $0 < p < 1$ and $p < q < \infty$ and without any conditions being imposed on the Banach space $\mathbf{B}$ we obtain in Theorem 2.1 necessary and sufficient conditions for $X \in SLLN(p,q)$.
- **(ii)** For $1 \leq q < \infty$ we obtain assuming the Banach space $\mathbf{B}$ is of stable type p where $1 < p < 2$ (Theorem 2.2) or $p = 1$ (Theorem 2.3) necessary and sufficient conditions for $X \in SLLN(p,q)$.

Theorems 1.2, 2.1, 2.2, and 2.3 are new results when $\mathbf{B} = \mathbb{R}$ (Theorem 2.4).

When $\mathbf{B} = \mathbb{R}$, necessary and sufficient conditions for $X \in SLLN(p,q)$ for the case where $0 < q < 1 \leq p < 2$ and for the case where $0 < q \leq p < 1$ remain open problems.

The plan of the paper is as follows. The main results are stated in Section 2 and they are proved in Section 3. In Section 4, three examples will be provided for illustrating the necessary and sufficient conditions obtained in this paper.

2. Statement of the main results

With the preliminaries accounted for, the main results may be stated.

Theorem 2.1. *Let $0 < p < 1$ and $p < q < \infty$. Let $\{X_n;\ n \geq 1\}$ be a sequence of independent copies of a $\mathbf{B}$-valued random variable X. Then we have the following two statements:*

(a) $X \in SLLN(p, q)$ *if and only if* $\mathbb{E}\|X\|^p < \infty$,

(b) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n} \mathbb{E}\left(\frac{\|S_n\|}{n^{1/p}}\right)^q < \infty$ *if and only if* $\mathbb{E}\|X\|^q < \infty$.

Let X be a $\mathbf{B}$-valued random variable. For each $n \geq 1$, we define the *quantile* u_n of order $1 - \frac{1}{n}$ of $\|X\|$ as follows:

$$u_n = \inf\left\{t:\ \mathbb{P}(\|X\| \leq t) > 1 - \frac{1}{n}\right\} = \inf\left\{t:\ \mathbb{P}(\|X\| > t) < \frac{1}{n}\right\}.$$

If $\mathbb{E}\|X\| < \infty$, then it is easy to show that $\lim_{n\to\infty} \frac{u_n}{n} = 0$.

Theorem 2.2. *Let $1 < p < 2$ and $1 \leq q < \infty$. Let $\mathbf{B}$ be a Banach space of stable type p. Let $\{X_n;\ n \geq 1\}$ be a sequence of independent copies of a $\mathbf{B}$-valued random variable X. Then*

$$X \in SLLN(p, q) \tag{2.1}$$

if and only if

$$\begin{cases} \mathbb{E}X = 0 \text{ and} \\ \begin{cases} \displaystyle\int_0^{\infty} \mathbb{P}^{q/p}\left(\|X\|^q > t\right) dt < \infty & \text{if } 1 \leq q < p, \\ \mathbb{E}\|X\|^p < \infty \text{ and } \displaystyle\sum_{n=1}^{\infty} \frac{\int_{\min\{u_n^p, n\}}^{n} \mathbb{P}\left(\|X\|^p > t\right) dt}{n} < \infty & \text{if } q = p, \\ \mathbb{E}\|X\|^p < \infty & \text{if } q > p. \end{cases} \end{cases} \tag{2.2}$$

Remark 2.1. When $q = 1$ and $\mathbf{B}$ is of stable p where $1 < p < 2$, Corollary 2.1 of Li, Qi, and Rosalsky [10] follows immediately from Theorems 1.2 and 2.2; that is, (1.1), (1.2), and (2.2) are equivalent.

Note by Lemma 5.6 of Li, Qi, and Rosalsky [10] that

$$\sum_{n=1}^{\infty} \frac{\int_{\min\{u_n^p, n\}}^{n} \mathbb{P}\left(\|X\|^p > t\right) dt}{n} < \infty \text{ whenever } \mathbb{E}\|X\|^p \ln^{\delta}(1+\|X\|) < \infty \text{ for some } \delta > 0.$$

Thus, for the interesting case $q = p$, Theorem 2.2 yields the following result.

Corollary 2.1. Let $1 < p < 2$ and let $\{X_n;\ n \geq 1\}$ be a sequence of independent copies of a $\mathbf{B}$-valued random variable X. If $\mathbf{B}$ is of stable type p, then

$X \in SLLN(p, p)$ whenever $\mathbb{E}X = 0$ and $\mathbb{E}\|X\|^p \ln^{\delta}(1 + \|X\|) < \infty$ for some $\delta > 0$.

For the case where $1 < p < 2$ and $1 \leq q < \infty$, combining Theorems 1.2 and 2.2, we immediately obtain necessary and sufficient conditions for (1.1) to hold assuming that $\mathbf{B}$ is of stable type p.

Corollary 2.2. Let $1 < p < 2$ and $1 \leq q < \infty$. Let X be a **B**-valued random variable. If **B** is of stable type p, then (1.1) holds if and only if

$$\begin{cases} \mathbb{E}X = 0 \text{ and} \\ \begin{cases} \displaystyle\int_0^\infty \mathbb{P}^{q/p}\left(\|X\|^q > t\right) dt < \infty & \text{if } 1 \leq q < p, \\ \mathbb{E}\|X\|^p \ln(1 + \|X\|) < \infty & \text{if } q = p, \\ \mathbb{E}\|X\|^q < \infty & \text{if } q > p. \end{cases} \end{cases}$$

Remark 2.2. For the case where $q = 1$, Corollary 2.2 above is Theorem 1.1 (i.e., Theorem 5 of Hechner and Heinkel [5]). Actually Corollary 2.2 for the case where $q = 1$ is somewhat stronger than Theorem 5 (necessity half) of Hechner and Heinkel [5] because $\mathbb{E}X = 0$ is an assumption in Theorem 5 of Hechner and Heinkel [5].

We now present necessary and sufficient conditions for (1.2) for the case where $p = 1$ and $1 \leq q < \infty$.

Theorem 2.3. *Let $1 \leq q < \infty$ and let* **B** *be a Banach space of stable type* 1*. Let $\{X_n;\ n \geq 1\}$ be a sequence of independent copies of a* **B***-valued random variable X. Then*

$$X \in SLLN(1, q) \tag{2.3}$$

if and only if

(2.4)

$$\begin{cases} \mathbb{E}\|X\| < \infty,\ \mathbb{E}X = 0, \textit{ and} \\ \begin{cases} \displaystyle\sum_{n=1}^\infty \frac{\|\mathbb{E}XI\{\|X\| \leq n\}\|}{n} < \infty \textit{ and} \sum_{n=1}^\infty \frac{\int_{\min\{u_n, n\}}^n \mathbb{P}(\|X\| > t)dt}{n} < \infty & \textit{if } q = 1, \\ \displaystyle\sum_{n=1}^\infty \frac{\|\mathbb{E}XI\{\|X\| \leq n\}\|^q}{n} < \infty & \textit{if } q > 1. \end{cases} \end{cases}$$

Remark 2.3. For the case where $q = 1$, Theorem 2.3 is Theorem 2.3 of Li, Qi, and Rosalsky [10].

By Lemmas 5.5 and 5.6 of Li, Qi, and Rosalsky [10], (2.4) holds whenever $\mathbb{E}X = 0$ and $\mathbb{E}\|X\| \ln(1 + \|X\|) < \infty$. Combining Theorems 1.2 and 2.3, we immediately have the following result.

Corollary 2.3. Let $1 \leq q < \infty$ and let **B** be a Banach space of stable type 1. Let $\{X_n;\ n \geq 1\}$ be a sequence of independent copies of a **B**-valued random variable X. Then (1.1) holds with $p = 1$ if and only if

$$\begin{cases} \mathbb{E}X = 0 \text{ and} \\ \begin{cases} \mathbb{E}\|X\| \ln(1 + \|X\|) < \infty & \text{if } q = 1, \\ \mathbb{E}\|X\|^q < \infty & \text{if } q > 1. \end{cases} \end{cases}$$

As a summary of our Theorems 1.2 and 2.1-2.3 and Corollaries 2.2 and 2.3, we now present the following theorem for a real-valued random variable X. For $q = 1$, the equivalence of (i) and (ii) has recently been obtained by Li, Qi, and Rosalsky [10], and for $1 = q < p < 2$, the equivalence of (iii) and (iv) is due to Hechner and Heinkel [5] (see Theorem 1.1 above) assuming that $\mathbb{E}X = 0$ for the implication ((iii) $\Rightarrow$ (iv)).

Theorem 2.4. *Let $0 < p < 2$ and $1 \le q < \infty$. Let $\{X_n;\ n \ge 1\}$ be a sequence of independent copies of a real-valued random variable X. The following two statements are equivalent:*

(i) $X \in SLLN(p, q)$,

(ii)
$$\begin{cases} \mathbb{E}X = 0 \textit{ and } \displaystyle\int_0^\infty \mathbb{P}^{q/p}\left(|X|^q > t\right) dt < \infty & \textit{if } 1 \le q < p < 2, \\ \mathbb{E}X = 0,\ \mathbb{E}|X|^p < \infty,\ \textit{and} & \\ \displaystyle\sum_{n=1}^{\infty} \frac{\int_{\min\{u_n^p, n\}}^{n} \mathbb{P}\left(|X|^p > t\right) dt}{n} < \infty & \textit{if } 1 < q = p < 2, \\ \mathbb{E}X = 0 \textit{ and } \mathbb{E}|X|^p < \infty & \textit{if } 1 < p < 2 \textit{ and } q > p, \\ \mathbb{E}X = 0,\ \displaystyle\sum_{n=1}^{\infty} \frac{\left|\mathbb{E}XI\left\{|X| \le n\right\}\right|}{n} < \infty,\ \textit{and} & \\ \displaystyle\sum_{n=1}^{\infty} \frac{\int_{\min\{u_n, n\}}^{n} \mathbb{P}\left(|X| > t\right) dt}{n} < \infty & \textit{if } q = p = 1, \\ \mathbb{E}X = 0 \textit{ and } \displaystyle\sum_{n=1}^{\infty} \frac{\left|\mathbb{E}XI\left\{|X| \le n\right\}\right|^q}{n} < \infty & \textit{if } p = 1 < q, \\ \mathbb{E}|X|^p < \infty & \textit{if } 0 < p < 1 \le q. \end{cases}$$

The following two statements are equivalent:

(iii) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n} \mathbb{E}\left(\frac{|S_n|}{n^{1/p}}\right)^q < \infty,$

(iv)
$$\begin{cases} \mathbb{E}X = 0 \textit{ and } \displaystyle\int_0^\infty \mathbb{P}^{q/p}\left(|X|^q > t\right) dt < \infty & \textit{if } 1 \le q < p < 2, \\ \mathbb{E}X = 0 \textit{ and } \mathbb{E}|X|^p \ln(1 + |X|) < \infty & \textit{if } 1 \le q = p < 2, \\ \mathbb{E}X = 0 \textit{ and } \mathbb{E}|X|^p < \infty & \textit{if } 1 \le p < 2 \textit{ and } q > p, \\ \mathbb{E}|X|^q < \infty & \textit{if } 0 < p < 1 \le q. \end{cases}$$

3. Proofs of Theorems 2.1 - 2.3

In this section we denote by C_k positive constants the precise values of which do not matter.

First we introduce some notation. Let $(a_k)_{1\le k\le n}$ be a finite sequence of real numbers and $(a_k^*)_{1\le k\le n}$ the nonincreasing rearrangement of the sequence $(|a_k|)_{1\le k\le n}$. For a given $r \ge 1$,

$$\|(a_k)_{1\le k\le n}\|_{r,\infty} = \sup_{1\le k\le n} k^{1/r} a_k^*$$

is called the *weak-ℓ_r norm* of the sequence $(a_k)_{1\le k\le n}$. Let $V_k, 1 \le k \le n$ be independent real-valued random variables. Then the remarkable Marcus-Pisier [14] inequality asserts that for all $r \ge 1$,

$$\mathbb{P}\left(\|(V_k)_{1\le k\le n}\|_{r,\infty} > u\right) \le \frac{2e}{u^r} \sup_{t>0} \left(t^r \sum_{k=1}^{n} \mathbb{P}\left(|V_k| > t\right)\right) \ \forall\, u > 0. \tag{3.1}$$

The original Marcus-Pisier [14] inequality involved the constant 262 instead of $2e$. The improved constant is due to J. Zinn (see Pisier [17, Lemma 4.11]).

Let X be a $\mathbf{B}$-valued random variable. For each $n \ge 1$, let the quantile u_n of order $1 - \frac{1}{n}$ of $\|X\|$ be defined as in Section 2. We then see that for every $q > 0$,

$$\inf\left\{t:\ \mathbb{P}\left(\|X\|^q \le t\right) > 1 - \frac{1}{n}\right\} = \inf\left\{t:\ \mathbb{P}\left(\|X\|^q > t\right) < \frac{1}{n}\right\} = u_n^q;$$

i.e., u_n^q is the quantile of order $1 - \frac{1}{n}$ of $\|X\|^q$. Let $\{X_n;\ n \ge 1\}$ be a sequence of independent copies of $\mathbf{B}$-valued variable X. Write, for $n \ge 1$,

$$S_n^{(1)} = \sum_{k=1}^{n} X_k I\{\|X_k\|^p \le k\},\ S_n^{(2)} = S_n - S_n^{(1)} = \sum_{k=1}^{n} X_k I\{\|X_k\|^p > k\},$$

$$U_n = \sum_{k=1}^{n} X_k I\{\|X_k\|^p \le n\},\ U_n^{(1)} = \sum_{k=1}^{n} X_k I\{\|X_k\| \le u_n\},\ \text{and } U_n^{(2)} = U_n - U_n^{(1)}.$$

Motivated by Lemma 1 of Hechner and Heinkel [5] and its proof, we establish the following result.

Lemma 3.1. *Let $1 < p < 2$ and $1 \le q < p$. Let $\mathbf{B}$ be a Banach space of stable type p. Then there exists a universal constant $c(p,q) > 0$ such that, for every finite sequence $V_k, 1 \le k \le n$ of independent $\mathbf{B}$-valued random variables with* $\max_{1\le k\le n} \mathbb{E}\|V_k\|^q < \infty$,

$$\mathbb{E}\left\|\sum_{k=1}^{n} \left(V_k - \mathbb{E}V_k\right)\right\|^q \le c(p,q) \left(\sup_{t>0} t^{p/q} \sum_{k=1}^{n} \mathbb{P}\left(\|V_k\|^q > t\right)\right)^{q/p}. \tag{3.2}$$

Remark 3.1. Clearly, if $q = 1$, then Lemma 3.1 is Lemma 1 of Hechner and Heinkel [5].

Proof of Lemma 3.1 Let $\{V_k';\ 1 \le k \le n\}$ be an independent copy of $\{V_k;\ 1 \le k \le n\}$ and let $\{R_k;\ 1 \le k \le n\}$ be a Rademacher sequence independent of $\{V_k, V_k';\ 1 \le k \le n\}$. Since $q \ge 1$, $g(x) = x^p,\ x \in [0,\infty)$ is a convex nonnegative function. Applying (2.5) of Ledoux and Talagrand [9, p. 46], we have that

$$\mathbb{E}\left\|\sum_{k=1}^{n} \left(V_k - \mathbb{E}V_k\right)\right\|^q \le \mathbb{E}\left\|\sum_{k=1}^{n} \left(V_k - V_k'\right)\right\|^q = \mathbb{E}\left\|\sum_{k=1}^{n} R_k\left(V_k - V_k'\right)\right\|^q \le 2^{q-1}\mathbb{E}\left\|\sum_{k=1}^{n} R_k V_k\right\|^q. \tag{3.3}$$

Since $\mathbf{B}$ is of stable type p with $1 \le p < 2$, the Maurey-Pisier [15] theorem asserts that it is also of stable type r for some $r > p$. Let $(A_k^*)_{1\le k\le n}$ be the nonincreasing rearrangement of $(\|V_k\|)_{1\le k\le n}$. Note that $r/q > 1$, $p/q > 1$ (since $1 \le q < p < r$), and $\mathbf{B}$ is also of Rademacher type r. We thus have that

$$
\begin{aligned}
\mathbb{E}\left\|\sum_{k=1}^{n} R_k V_k\right\|^q &\le \mathbb{E}\left(\mathbb{E}\left(\left\|\sum_{k=1}^{n} R_k V_k\right\|^r \middle| V_1, ..., V_n\right)\right)^{1/(r/q)} \\
&\le C_1 \mathbb{E}\left(\sum_{k=1}^{n} \|V_k\|^r\right)^{q/r} \\
&= C_1 \mathbb{E}\left(\sum_{k=1}^{n} \left(k^{r/p}\left(A_k^*\right)^r\right) k^{-r/p}\right)^{q/r} \qquad (3.4) \\
&\le C_1 \mathbb{E}\left(\left(\sup_{1\le k\le n} k^{q/p}(A_k^*)^q\right)\left(\sum_{k=1}^{n} k^{-r/p}\right)^{q/r}\right) \\
&= C_2 \mathbb{E}\left\|(\|V_k\|^q)_{1\le k\le n}\right\|_{p/q,\infty}.
\end{aligned}
$$

Write $\Delta = \sup_{t>0} t^{p/q} \sum_{k=1}^{n} \mathbb{P}\left(\|V_k\|^q > t\right)$. Using the Marcus-Pisier [14] inequality (3.1), we have that

(3.5)

$$
\begin{aligned}
\mathbb{E}\left\|(\|V_k\|^q)_{1\le k\le n}\right\|_{p/q,\infty} &= \left(\int_0^{\Delta^{q/p}} + \int_{\Delta^{q/p}}^{\infty}\right) \mathbb{P}\left(\left\|(\|V_k\|^q)_{1\le k\le n}\right\|_{p/q,\infty} > t\right) dt \\
&\le \Delta^{q/p} + \int_{\Delta^{q/p}}^{\infty} \frac{2e\Delta}{t^{p/q}} dt \\
&= \left(1 + \frac{2qe}{p-q}\right)\Delta^{q/p}.
\end{aligned}
$$

Now (3.2) follows from (3.3), (3.4), and (3.5). □

The following nice result is Proposition 3 of Hechner and Heinkel [5].

Lemma 3.2. (Hechner and Heinkel [5]). *Let $p > 1$ and let $\{X_n;\ n \ge 1\}$ be a sequence of independent copies of a $\mathbf{B}$-valued random variable X. Then the following three statements are equivalent:*

- **(i)** $\int_0^{\infty} \mathbb{P}^{1/p}(\|X\| > t)dt < \infty;$
- **(ii)** $\sum_{n=1}^{\infty} \frac{u_n}{n^{1+1/p}} < \infty;$
- **(iii)** $\sum_{n=1}^{\infty} \frac{1}{n^{1+1/p}} \mathbb{E}\left(\max_{1\le k\le n} \|X_k\|\right) < \infty.$

The next lemma and its proof are similar to Lemma 3 of Hechner and Heinkel [5] and its proof, respectively.

Lemma 3.3. *Let $1 \le q < p < 2$. Let X be a $\mathbf{B}$-valued random variable with*

$$\int_0^\infty \mathbb{P}^{q/p}\left(\|X\|^q > t\right) dt < \infty. \tag{3.6}$$

If $\mathbf{B}$ is a Banach space of Rademacher type q, then

$$\sum_{n=1}^\infty \frac{\mathbb{E}\left\|\left(S_n - U_n^{(1)}\right) - \mathbb{E}\left(S_n - U_n^{(1)}\right)\right\|^q}{n^{1+q/p}} < \infty. \tag{3.7}$$

Proof. Let $f_q(t) = \mathbb{P}\left(\|X\|^q > t\right),\ t \ge 0$. Since $\mathbf{B}$ is a Banach space of Rademacher type q and

$$\left(S_n - U_n^{(1)}\right) - \mathbb{E}\left(S_n - U_n^{(1)}\right) = \sum_{k=1}^n \left(X_k I\{\|X_k\| > u_n\} - \mathbb{E}XI\{\|X\| > u_n\}\right),\ n \ge 1,$$

by Theorem 2.1 of Hoffmann-Jørgensen and Pisier [7], we have that

$$\begin{aligned} &\mathbb{E}\left\|\left(S_n - U_n^{(1)}\right) - \mathbb{E}\left(S_n - U_n^{(1)}\right)\right\|^q \\ &\le C_3 n \mathbb{E}\left\|XI\{\|X\| > u_n\} - \mathbb{E}XI\{\|X\| > u_n\}\right\|^q \\ &\le C_4 n \mathbb{E}\left(\|X\|^q I\left\{\|X\|^q > u_n^q\right\}\right) \\ &\le C_4\left(u_n^q + n\int_{u_n^q}^\infty f_q(t)dt\right). \end{aligned} \tag{3.8}$$

Set $p_1 = p/q$, $Y = \|X\|^q$, and $u_{n,q} = u_n^q$, $n \ge 1$. Noting that $p_1 > 1$ (since $1 \le q < p < 2$), by Lemma 3.2 (i.e., Proposition 3 of Hechner and Heinkel [5]), it follows from (3.6) that

$$\sum_{n=1}^\infty \frac{u_n^q}{n^{1+q/p}} = \sum_{n=1}^\infty \frac{u_{n,q}}{n^{1+1/p_1}} < \infty. \tag{3.9}$$

Also (3.6) implies that

$$\begin{aligned} \sum_{n=1}^\infty \frac{1}{n^{q/p}} \int_{u_n^q}^\infty f_q(t)dt &= \sum_{n=1}^\infty n^{-1/p_1} \sum_{j=n}^\infty \int_{u_{j,q}}^{u_{j+1,q}} f_q(t)dt \\ &= \sum_{j=1}^\infty \left(\int_{u_{j,q}}^{u_{j+1,q}} f_q(t)dt\right) \sum_{n=1}^j n^{-1/p_1} \\ &\le C_5 \sum_{j=1}^\infty \left(\int_{u_{j,q}}^{u_{j+1,q}} f_q^{1/p_1}(t)dt\right) \frac{j^{1-1/p_1}}{j^{1-1/p_1}} \\ &\le C_5 \int_0^\infty f_q^{1/p_1}(t)dt \\ &= C_5 \int_0^\infty \mathbb{P}^{q/p}\left(\|X\|^q > t\right) dt \\ &< \infty. \end{aligned} \tag{3.10}$$

The conclusion (3.7) follows from (3.8), (3.9), and (3.10). □

The proof of the next lemma is similar to that of Lemma 4 of Hechner and Heinkel [5] and Lemma 5.3 of Li, Qi, and Rosalsky [10] and it contains a nice application of Lemma 3.1 above.

Lemma 3.4. *Let $1 \le q \le p < 2$. Let X be a $\mathbf{B}$-valued random variable with (3.6). If $\mathbf{B}$ is a Banach space of stable type p, then*

$$\sum_{n=1}^{\infty} \frac{\mathbb{E}\left\|U_n^{(1)} - \mathbb{E}U_n^{(1)}\right\|^q}{n^{1+q/p}} < \infty. \tag{3.11}$$

Remark 3.2. Note that $\mathbb{E}\|X\|^q = \int_0^\infty \mathbb{P}\left(\|X\|^q > t\right) dt$. Thus for $q = p$, (3.6) holds if and only if $\mathbb{E}\|X\|^p < \infty$. By Lemma 3.4, if $\mathbf{B}$ is a Banach space of stable type $p \in [1, 2)$, then

$$\sum_{n=1}^{\infty} \frac{\mathbb{E}\left\|U_n^{(1)} - \mathbb{E}U_n^{(1)}\right\|^p}{n^2} < \infty \text{ whenever } \mathbb{E}\|X\|^p < \infty. \tag{3.12}$$

Proof of Lemma 3.4 Since $\mathbf{B}$ is of stable type p, the Maurey-Pisier [15] theorem asserts that it is also of stable type r for some $r > p$. Applying Lemma 3.1, there exists a universal constant $0 < c(r, q) < \infty$ such that

$$\begin{aligned}\mathbb{E}\left\|U_n^{(1)} - \mathbb{E}U_n^{(1)}\right\|^q &\le c(r,q)\left(\sup_{t>0} t^{r/q}\sum_{k=1}^{n}\mathbb{P}\left(\|X_k\|^q I\{\|X_k\| \le u_n\} > t\right)\right)^{q/r} \\ &\le c(r,q)\left(n\sup_{0\le t\le u_n^q} t^{r/q}\mathbb{P}\left(\|X\|^q > t\right)\right)^{q/r}, \ n \ge 1.\end{aligned}$$

It is easy to see that for all $x > 0$,

$$\left(\int_0^x \mathbb{P}^{q/r}\left(\|X\|^q > t\right) dt\right)^{r/q} \ge \left(\int_0^x \mathbb{P}^{q/r}\left(\|X\|^q > x\right) dt\right)^{r/q} = x^{r/q}\mathbb{P}\left(\|X\|^q > x\right).$$

We thus have that

$$\begin{aligned}\mathbb{E}\left\|U_n^{(1)} - \mathbb{E}U_n^{(1)}\right\|^q &\le c(r,q)\left(n\sup_{0\le t\le u_n^q} t^{r/q}\mathbb{P}\left(\|X\|^q > t\right)\right)^{q/r} \\ &\le c(r,q)n^{q/r}\int_0^{u_n^q} \mathbb{P}^{q/r}\left(\|X\|^q > t\right) dt, \ n \ge 1.\end{aligned}$$

Let $u_0 = 0$ and note that $\mathbb{P}(\|X\|^q > t) \geq 1/k$ for $t \in [u_{k-1}^q, u_k^q),\ k \geq 1$. It follows that

$$\begin{aligned}
\sum_{n=1}^{\infty} \frac{\mathbb{E}\left\|U_n^{(1)} - \mathbb{E}U_n^{(1)}\right\|^q}{n^{1+q/p}} &\leq c(r,q) \sum_{n=1}^{\infty} \frac{1}{n^{1+q/p-q/r}} \int_0^{u_n^q} \mathbb{P}^{q/r}\left(\|X\|^q > t\right) dt \\
&= c(r,q) \sum_{k=1}^{\infty} \left(\sum_{n=k}^{\infty} \frac{1}{n^{1+q/p-q/r}} \right) \int_{u_{k-1}^q}^{u_k^q} \mathbb{P}^{q/r}\left(\|X\|^q > t\right) dt \\
&\leq C_6 \sum_{k=1}^{\infty} \frac{1}{k^{q/p-q/r}} \int_{u_{k-1}^q}^{u_k^q} \mathbb{P}^{q/r}\left(\|X\|^q > t\right) dt \\
&\leq C_6 \sum_{k=1}^{\infty} \int_{u_{k-1}^q}^{u_k^q} \mathbb{P}^{q/p}\left(\|X\|^q > t\right) dt \\
&= C_6 \int_0^{\infty} \mathbb{P}^{q/p}\left(\|X\|^q > t\right) dt < \infty
\end{aligned}$$

proving (3.11) and completing the proof of Lemma 3.4. □

Lemma 3.5. *Let $1 \leq p < 2$ and let X be a $\mathbf{B}$-valued random variable with $\mathbb{E}\|X\|^p < \infty$. Then*

$$\sum_{n=1}^{\infty} \frac{1}{n^2} \left(\sum_{k=1}^{n} \mathbb{E}\|X\| I\left\{k < \|X\|^p \leq n\right\} \right)^p < \infty, \tag{3.13}$$

$$\sum_{n=1}^{\infty} \frac{u_n^p}{n^2} < \infty, \tag{3.14}$$

and for every $\delta > 0$,

$$\sum_{n=1}^{\infty} \frac{\mathbb{E}\|X\|^{p+\delta} I\left\{\|X\|^p \leq n\right\}}{n^{1+\delta/p}} < \infty. \tag{3.15}$$

Furthermore, if $p > 1$ then

$$\sum_{n=1}^{\infty} \frac{\left(\mathbb{E}\|X\| I\left\{\|X\|^p > n\right\}\right)^p}{n^{2-p}} < \infty. \tag{3.16}$$

Remark 3.3. For $p = 1$, (3.13) and (3.14) together are Lemma 5.1 of Li, Qi, and Rosalsky [10].

Proof of Lemma 3.5 Since u_n^p is the quantile of order $1 - \frac{1}{n}$ of $\|X\|^p$, (3.14) immediately follows from the second half of Lemma 5.1 of Li, Qi, and Rosalsky [10].

The proof of (3.15) is easy and we leave it to the reader.

We now show that $\mathbb{E}\|X\|^p < \infty$ implies (3.13). For $n \geq 2$, let

$$\Lambda_n = \sum_{k=2}^{n} k\mathbb{P}\left(k-1 < \|X\|^p \leq k\right),\ \lambda_{n,j} = \frac{j\mathbb{P}\left(j-1 < \|X\|^p \leq j\right)}{\Lambda_n},\ 2 \leq j \leq n.$$

Clearly

$$\lambda_{n,j} \geq 0,\ 2 \leq j \leq n,\ \sum_{j=2}^{n} \lambda_{n,j} = 1,\ \text{and}\ \Lambda_n \leq \mathbb{E}\|X\|^p + 1 < \infty,\ n \geq 2.$$

Note that the function $\phi(t) = t^p$ is convex on $[0, \infty)$ and

$$\begin{aligned}
\|X\| \sum_{k=1}^{n} I\{k < \|X\|^p \leq n\} &= \sum_{k=1}^{n} \sum_{j=k+1}^{n} \|X\| I\{j-1 < \|X\|^p \leq j\} \\
&\leq \sum_{j=2}^{n} j^{1+1/p} I\{j-1 < \|X\|^p \leq j\},\ n \geq 2.
\end{aligned}$$

We thus have that

$$\begin{aligned}
\left(\sum_{k=1}^{n} \mathbb{E}\|X\| I\{k < \|X\|^p \leq n\}\right)^p &= \left(\mathbb{E}\left(\|X\| \sum_{k=1}^{n} I\{k < \|X\|^p \leq n\}\right)\right)^p \\
&\leq \left(\mathbb{E}\left(\sum_{j=2}^{n} j^{1+1/p} I\{j-1 < \|X\|^p \leq j\}\right)\right)^p \\
&= \Lambda_n^p \left(\sum_{j=2}^{n} j^{1/p} \lambda_{n,j}\right)^p \\
&\leq \Lambda_n^p \sum_{j=2}^{n} \lambda_{n,j} \left(j^{1/p}\right)^p \\
&= \Lambda_n^{p-1} \sum_{j=2}^{n} j^2 \mathbb{P}(j-1 < \|X\|^p \leq j) \\
&\leq C_7 \sum_{j=2}^{n} j^2 \mathbb{P}(j-1 < \|X\|^p \leq j),\ n \geq 2.
\end{aligned}$$

It now is easy to see that

$$\begin{aligned}
\sum_{n=1}^{\infty} \frac{1}{n^2} \sum_{j=2}^{n} j^2 \mathbb{P}(j-1 < \|X\|^p \leq j) &= \sum_{j=2}^{\infty} \left(\sum_{n=j}^{\infty} \frac{1}{n^2}\right) j^2 \mathbb{P}(j-1 < \|X\|^p \leq j) \\
&\leq C_8 \sum_{j=2}^{\infty} j \mathbb{P}(j-1 < \|X\|^p \leq j) \\
&\leq C_8 (\mathbb{E}\|X\|^p + 1) < \infty
\end{aligned}$$

thereby proving (3.13).

We now prove (3.16). Note that for $n \geq 1$,

$$\begin{aligned}\mathbb{E}\|X\| I\{\|X\|^p > n\} &\leq \sum_{j=n+1}^{\infty} j^{1/p} \mathbb{P}\left(j-1 < \|X\|^p \leq j\right) \\ &= \sum_{j=n+1}^{\infty} j^{1/p-1}\left(j\mathbb{P}\left(j-1 < \|X\|^p \leq j\right)\right)\end{aligned}$$

and

$$\sum_{j=n}^{\infty} j\mathbb{P}\left(j-1 < \|X\|^p \leq j\right) \leq \mathbb{E}\|X\|^p + 1.$$

Thus, by the same arguments used in proving (3.13), we have that

$$\left(\mathbb{E}\|X\| I\{\|X\|^p > n\}\right)^p \leq C_9 \sum_{j=n}^{\infty} j^{1-p}\left(j\mathbb{P}\left(j-1 < \|X\|^p \leq j\right)\right), \ n \geq 1.$$

Since $p > 1$, we get that

$$\begin{aligned}\sum_{n=1}^{\infty} \frac{\left(\mathbb{E}\|X\| I\{\|X\|^p > n\}\right)^p}{n^{2-p}} &\leq C_9 \sum_{n=1}^{\infty} \frac{1}{n^{2-p}} \sum_{j=n}^{\infty} j^{1-p}\left(j\mathbb{P}\left(j-1 < \|X\|^p \leq j\right)\right) \\ &\leq C_{10} \sum_{j=1}^{\infty} j\mathbb{P}\left(j-1 < \|X\|^p \leq j\right) \\ &\leq C_{10}\left(\mathbb{E}\|X\|^p + 1\right) < \infty\end{aligned}$$

proving (3.16). □

Lemma 3.6. *Let $1 \leq p < 2$ and let $\{X_n;\ n \geq 1\}$ be a sequence of independent copies of a $\mathbf{B}$-valued random variable X with $\mathbb{E}X = 0$ and $\mathbb{E}\|X\|^p < \infty$. If $\mathbf{B}$ is a Banach space of Rademacher type p, then*

(3.17)

$$\sum_{n=1}^{\infty} \frac{1}{n} \mathbb{E}\left(\frac{\left\|S_n^{(1)} - \mathbb{E}S_n^{(1)}\right\|^p}{n}\right) < \infty \text{ if and only if } \sum_{n=1}^{\infty} \frac{1}{n} \mathbb{E}\left(\frac{\|U_n - \mathbb{E}U_n\|^p}{n}\right) < \infty.$$

Proof Note that, for $n \geq 1$,

$$\left(S_n^{(1)} - \mathbb{E}S_n^{(1)}\right) - \left(U_n - \mathbb{E}U_n\right) = \sum_{k=1}^{n}\left(X_k I\{k < \|X_k\|^p \leq n\} - \mathbb{E}XI\{k < \|X\|^p \leq n\}\right).$$

Then since $\mathbf{B}$ is a Banach space of Rademacher type p, by Theorem 2.1 of Hoffmann-Jørgensen and Pisier [7], we have that

$$\begin{aligned}&\mathbb{E}\left\|\left(S_n^{(1)} - \mathbb{E}S_n^{(1)}\right) - \left(U_n - \mathbb{E}U_n\right)\right\|^p \\ &\leq C_{11} \sum_{k=1}^{n} \mathbb{E}\left\|XI\{k < \|X\|^p \leq n\} - \mathbb{E}XI\{k < \|X\|^p \leq n\}\right|^p \\ &\leq C_{12} \sum_{k=1}^{n} \mathbb{E}\|X\|^p I\{k < \|X\|^p \leq n\}, \ n \geq 1.\end{aligned}$$

Let $Y = \|X\|^p$. Then it follows from $\mathbb{E}Y < \infty$ (since $\mathbb{E}\|X\|^p < \infty$) and the first conclusion of Lemma 3.5 (i.e., (3.13)) that

$$\sum_{n=1}^{\infty} \frac{1}{n} \mathbb{E}\left(\frac{\left\| \left(S_n^{(1)} - \mathbb{E}S_n^{(1)}\right) - (U_n - \mathbb{E}U_n) \right\|^p}{n} \right) \leq \sum_{n=1}^{\infty} \frac{C_{12}}{n^2} \sum_{k=1}^{n} \mathbb{E}YI\{k < Y \leq n\} < \infty,$$

which yields (3.17). □

Proof of Theorem 2.1 To prove Theorem 2.1, we make the following simple observation. Let $0 < p < q \leq 1$. Let $\{X_n;\ n \geq 1\}$ be a sequence of independent copies of a **B**-valued random variable X with $\mathbb{E}\|X\|^p < \infty$. Set $p_1 = p/q$, $Y = \|X\|^q,\ Y_n = \|X_n\|^q,\ n \geq 1$. Then $0 < p_1 < 1$ and $\mathbb{E}Y^{p_1} < \infty$, and

$$\sum_{n=1}^{\infty} \frac{1}{n} \left(\frac{\|S_n\|}{n^{1/p}} \right)^q \leq \sum_{n=1}^{\infty} \frac{\sum_{k=1}^{n} \|X_k\|^q}{n^{1+q/p}} = \sum_{k=1}^{\infty} \sum_{n=k}^{\infty} \frac{\|X_k\|^q}{n^{1+q/p}} \leq C_{13} \sum_{k=1}^{\infty} k^{-1/p_1} Y_k < \infty \text{ a.s.}$$

(see Theorem 5.1.3 in Chow and Teicher [2, p. 118]). Theorem 2.1 follows immediately from this observation together with Theorem 1.2 and Remark 1.2. □

Proof of Theorem 2.2 (**Sufficiency**) Firstly we consider the case where $1 \leq q < p < 2$. Since $\mathbb{E}X = 0$, we see that

$$S_n = \left(U_n^{(1)} - \mathbb{E}U_n^{(1)}\right) + \left((S_n - U_n^{(1)}) - \mathbb{E}(S_n - U_n^{(1)})\right),\ n \geq 1$$

so that, by Lemmas 3.3 and 3.4, (2.2) ensures (1.1) which implies (2.1).

Secondly we consider the case where $1 < p < q$. Since **B** is of stable type p, the Maurey-Pisier [15] theorem asserts that it is also of stable type $p + \delta$ for some $0 < \delta < q - p$. By Remark 1.2, (2.1) holds if we can show that

$$X \in SLLN(p, p+\delta);\ \text{i.e.,} \sum_{n=1}^{\infty} \frac{1}{n} \left(\frac{\|S_n\|}{n^{1/p}} \right)^{p+\delta} < \infty \text{ a.s.} \tag{3.18}$$

Since $\mathbb{E}X = 0$, we have that, for $n \geq 1$,

$$\begin{aligned} S_n &= \sum_{k=1}^{n} X_k I\{\|X_k\|^p \leq n\} + \sum_{k=1}^{n} X_k I\{\|X_k\|^p > n\} \\ &= (U_n - \mathbb{E}U_n) - n\mathbb{E}XI\{\|X\|^p > n\} + \sum_{k=1}^{n} X_k I\{\|X_k\|^p > n\}. \end{aligned} \tag{3.19}$$

It is easy to see that

$$\left\{ \max_{1 \leq k \leq n} \|X_k\|^p > n \text{ i.o.}(n) \right\} = \{\|X_n\|^p > n \text{ i.o.}(n)\}.$$

Since $\{X_n;\ n \geq 1\}$ is a sequence of independent copies of **B**-valued random variable X with $\mathbb{E}\|X\|^p < \infty$, it follows from the Borel-Cantelli lemma that

$$\mathbb{P}\left(\|X_n\|^p > n \text{ i.o.}(n)\right) = 0$$

and hence

$$\mathbb{P}\left(\max_{1 \leq k \leq n} \|X_k\|^p > n \text{ i.o.}(n) \right) = 0, \tag{3.20}$$

which ensures that

$$\sum_{n=1}^{\infty} \frac{1}{n} \left(\frac{\|\sum_{k=1}^{n} X_k I\{\|X_k\|^p > n\}\|}{n^{1/p}} \right)^{p+\delta} < \infty \text{ a.s.} \tag{3.21}$$

Note that $1 < p < 2$ and $\mathbb{E}\|X\|^p < \infty$ imply that

$$\frac{n\mathbb{E}\|X\|I\{\|X\|^p > n\}}{n^{1/p}} \leq \frac{n^{1/p}\mathbb{E}\|X\|^p}{n^{1/p}} = \mathbb{E}\|X\|^p, \ n \geq 1.$$

Thus, by (3.16) of Lemma 3.5, we have that

$$\begin{aligned} &\sum_{n=1}^{\infty} \frac{1}{n} \left(\frac{\|n\mathbb{E}XI\{\|X\|^p > n\}\|}{n^{1/p}} \right)^{p+\delta} \\ &= \sum_{n=1}^{\infty} \frac{1}{n} \left(\frac{\|n\mathbb{E}XI\{\|X\|^p > n\}\|}{n^{1/p}} \right)^{p} \left(\frac{\|n\mathbb{E}XI\{\|X\|^p > n\}\|}{n^{1/p}} \right)^{\delta} \\ &\leq (\mathbb{E}\|X\|^p)^{\delta} \sum_{n=1}^{\infty} \frac{(\mathbb{E}\|X\|I\{\|X\|^p > n\})^p}{n^{2-p}} \\ &< \infty. \end{aligned} \tag{3.22}$$

Since **B** is also of Rademacher type $p + \delta$, by Theorem 2.1 of Hoffmann-Jørgensen and Pisier [7], we get that

$$\begin{aligned} \mathbb{E}\|U_n - \mathbb{E}U_n\|^{p+\delta} &\leq C_{14} n \mathbb{E}\|XI\{\|X\|^p > n\} - \mathbb{E}XI\{\|X\|^p > n\}\|^{p+\delta} \\ &\leq C_{15}\mathbb{E}\|X\|^{p+\delta} I\{\|X\|^p > n\}, \ n \geq 1. \end{aligned}$$

Thus, by (3.15) of Lemma 3.5, we have that

$$\sum_{n=1}^{\infty} \frac{1}{n} \mathbb{E} \left(\frac{\|U_n - \mathbb{E}U_n\|}{n^{1/p}} \right)^{p+\delta} \leq C_{15} \sum_{n=1}^{\infty} \frac{\mathbb{E}\|X\|^{p+\delta} I\{\|X\|^p > n\}}{n^{1+\delta/p}} < \infty$$

and hence

$$\sum_{n=1}^{\infty} \frac{1}{n} \left(\frac{\|U_n - \mathbb{E}U_n\|}{n^{1/p}} \right)^{p+\delta} < \infty \text{ a.s.},$$

which, together with (3.19), (3.21), and (3.22), ensures (3.18).

Lastly we consider the case where $1 < q = p < 2$. Since $\mathbb{E}\|X\|^p < \infty$, we have that

$$\lim_{n\to\infty} \frac{u_n^p}{n} = 0; \text{ i.e., } \lim_{n\to\infty} \frac{u_n}{n^{1/p}} = 0.$$

Hence we can assume, without loss of generality, that $u_n < n^{1/p}$ for all $n \geq 1$. Since $\mathbb{E}X = 0$, we have that, for $n \geq 1$,

(3.23)

$$\begin{aligned} S_n &= \sum_{k=1}^{n} X_k I\{\|X_k\| \leq u_n\} + \sum_{k=1}^{n} X_k I\{u_n < \|X_k\| \leq n^{1/p}\} + \sum_{k=1}^{n} X_k I\{\|X_k\| > n^{1/p}\} \\ &= \left(U_n^{(1)} - \mathbb{E}U_n^{(1)}\right) + \left(U_n^{(2)} - \mathbb{E}U_n^{(2)}\right) - n\mathbb{E}XI\left\{\|X\|^p > n\right\} + \sum_{k=1}^{n} X_k I\left\{\|X_k\|^p > n\right\}. \end{aligned}$$

Since (3.20) follows from $\mathbb{E}\|X\|^p < \infty$, we see that

$$\sum_{n=1}^{\infty} \frac{1}{n} \left(\frac{\|\sum_{k=1}^{n} X_k I\{\|X_k\|^p > n\}\|}{n^{1/p}} \right)^p < \infty \text{ a.s.} \tag{3.24}$$

Since $p > 1$ and $\mathbb{E}\|X\|^p < \infty$, it follows from (3.16) of Lemma 3.5 that

$$\sum_{n=1}^{\infty} \frac{1}{n} \left(\frac{\|n\mathbb{E}XI\{\|X\|^p > n\}\|}{n^{1/p}} \right)^p \leq \sum_{n=1}^{\infty} \frac{(\mathbb{E}\|X\|I\{\|X\|^p > n\})^p}{n^{2-p}} < \infty. \tag{3.25}$$

Since $\mathbb{E}\|X\|^p < \infty$ and $\mathbf{B}$ is a Banach space of stable type $p \in (1, 2)$, by Remark 3.2, (3.12) holds, which ensures that

$$\sum_{n=1}^{\infty} \frac{1}{n} \left(\frac{\left\|U_n^{(1)} - \mathbb{E}U_n^{(1)}\right\|}{n^{1/p}} \right)^p < \infty \text{ a.s.} \tag{3.26}$$

Since $\mathbf{B}$ is also a Banach space of Rademacher type p, by Theorem 2.1 of Hoffmann-Jørgensen and Pisier [7], we have that, for all $n \geq 1$,

$$\begin{aligned}
&\mathbb{E}\|U_n^{(2)} - \mathbb{E}U_n^{(2)}\|^p \\
&\leq C_{16} \sum_{k=1}^{n} \mathbb{E} \left\| X_k I\left\{u_n < \|X_k\| \leq n^{1/p}\right\} - \mathbb{E}\left(XI\left\{u_n < \|X\| \leq n^{1/p}\right\}\right)\right\|^p \\
&\leq 2C_{16} n \mathbb{E}\left(\|X\|^p I\{u_n^p < \|X\|^p \leq n\}\right) \\
&\leq 2C_{16} n u_n^p \mathbb{P}\left(\|X\|^p > u_n^p\right) + 2C_{16} n \int_{u_n^p}^{n} \mathbb{P}\left(\|X\|^p > t\right) dt \\
&\leq 2C_{16} u_n^p + 2C_{16} n \int_{u_n^p}^{n} \mathbb{P}\left(\|X\|^p > t\right) dt.
\end{aligned}$$

Now (3.14) holds by Lemma 3.5. Thus it follows from (3.14) and (2.2) that

$$\sum_{n=1}^{\infty} \frac{\mathbb{E}\|U_n^{(2)} - \mathbb{E}U_n^{(2)}\|^p}{n^2} < \infty,$$

which ensures that

$$\sum_{n=1}^{\infty} \frac{\|U_n^{(2)} - \mathbb{E}U_n^{(2)}\|^p}{n^2} < \infty \text{ a.s.} \tag{3.27}$$

Combining (3.23)-(3.27), we conclude that (2.1) holds for $q = p$. The proof of the sufficiency half of Theorem 2.2 is complete. □

Proof of Theorem 2.2 (**Necessity**) For the case where $q \neq p$, by Theorem 1.2, we see that (2.2) follows immediately from (2.1).

We now consider the case where $q = p$. By Theorem 1.2, (2.1) implies that $\mathbb{E}X = 0$ and $\mathbb{E}\|X\|^p < \infty$. Hence we can assume, without loss of generality, that $u_n^p < n$ for all $n \geq 1$. We thus only need to show that (2.1) (with $q = p$) implies that

$$\sum_{n=1}^{\infty} \frac{\int_{u_n^p}^{n} \mathbb{P}\left(\|X\|^p > t\right) dt}{n} < \infty. \tag{3.28}$$

To see this, let $\{X',\ X'_n;\ n \geq 1\}$ be an independent copy of $\{X,\ X_n;\ n \geq 1\}$. Let

$$V_n = \left(X_n I\left\{\|X_n\|^p \leq n\right\} - X'_n I\left\{\|X'_n\|^p \leq n\right\}\right) \text{ and } \hat{S}_n^{(1)} = \sum_{k=1}^{n} V_k,\ n \geq 1.$$

Then $\{V_n;\ n \geq 1\}$ is a sequence of independent symmetric $\mathbf{B}$-valued random variables. By the Borel-Cantelli lemma, it follows from $\mathbb{E}\|X\|^p < \infty$ that

$$\mathbb{P}\left(\|X_n\|^p > n \text{ i.o.}(n)\right) = 0,$$

which ensures that

$$S_n^{(2)} = \sum_{k=1}^{n} X_k I\left\{\|X_k\|^p > k\right\} = O(1) \text{ a.s. as } n \to \infty$$

and hence

$$\sum_{n=1}^{\infty} \frac{1}{n}\left(\frac{\|S_n^{(2)}\|}{n^{1/p}}\right)^p = \sum_{n=1}^{\infty} \frac{\|S_n^{(2)}\|^p}{n^2} < \infty \text{ a.s.}$$

Note that $\|S_n^{(1)}\| \leq \|S_n\| + \|S_n^{(2)}\|$, $n \geq 1$. It thus follows from (2.1) (with $q = p$) that

$$\sum_{n=1}^{\infty} \frac{\|S_n^{(1)}\|^p}{n^2} < \infty \text{ a.s.} \tag{3.29}$$

and hence

$$\sum_{n=1}^{\infty} \frac{\|\hat{S}_n^{(1)}\|^p}{n^2} < \infty \text{ a.s.} \tag{3.30}$$

Let $a_n = 1/n^2,\ n \geq 1$. Then

$$b_n = \sum_{k=n}^{\infty} a_k = \sum_{k=n}^{\infty} \frac{1}{n^2} \leq \frac{2}{n},\ n \geq 1$$

and hence

$$\sup_{n \geq 1} b_n \|V_n\|^p \leq \sup_{n \geq 1} \frac{2}{n}\left(2n^{1/p}\right)^p = 2^{p+1} \text{ a.s.}$$

We thus have that

$$\mathbb{E}\left(\sup_{n \geq 1} b_n \|V_n\|^p\right) < \infty. \tag{3.31}$$

By Theorem 2.4 of Li, Qi, and Rosalsky [11], we conclude from (3.30) and (3.31) that

$$\sum_{n=1}^{\infty} \frac{\mathbb{E}\|\hat{S}_n^{(1)}\|^p}{n^2} < \infty;$$

that is,

$$\mathbb{E}\left(\sum_{n=1}^{\infty} \frac{\|\hat{S}_n^{(1)}\|^p}{n^2}\right) < \infty. \tag{3.32}$$

By Lemma 3.4 of Li, Qi, and Rosalsky [11], it follows from (3.29) and (3.32) that

$$\mathbb{E}\left(\sum_{n=1}^{\infty} \frac{\|S_n^{(1)}\|^p}{n^2}\right) < \infty;$$

that is,

$$\sum_{n=1}^{\infty} \frac{\mathbb{E}\|S_n^{(1)}\|^p}{n^2} < \infty. \tag{3.33}$$

Since $1 < p < 2$, applying (2.5) of Ledoux and Talagrand [9, p. 46], (3.33) ensures that

$$\sum_{n=1}^{\infty} \frac{\|\mathbb{E}S_n^{(1)}\|^p}{n^2} < \infty$$

which, together with (3.33), gives

$$\sum_{n=1}^{\infty} \frac{\mathbb{E}\|S_n^{(1)} - \mathbb{E}S_n^{(1)}\|^p}{n^2} < \infty.$$

By Lemma 3.6, this is equivalent to

$$\sum_{n=1}^{\infty} \frac{\mathbb{E}\|U_n - \mathbb{E}U_n\|^p}{n^2} < \infty. \tag{3.34}$$

Since $\|U_n^{(2)} - \mathbb{E}U_n^{(2)}\| \leq \|U_n - \mathbb{E}U_n\| + \|U_n^{(1)} - \mathbb{E}U_n^{(1)}\|,\ n \geq 1$ and $\mathbf{B}$ is of stable type p where $1 < p < 2$, it follows from Remark 3.2 and (3.34) that

$$\sum_{n=1}^{\infty} \frac{\mathbb{E}\|U_n^{(2)} - \mathbb{E}U_n^{(2)}\|^p}{n^2} < \infty. \tag{3.35}$$

By Lemma 3.1 **(ii)** of Li, Qi, and Rosalsky [10],

$$\begin{aligned}
&\mathbb{E} \max_{1\leq k\leq n} \|X_k I\{u_n^p < \|X_k\|^p \leq n\} - \mathbb{E}XI\{u_n^p < \|X\|^p \leq n\}\|^p \\
&= \mathbb{E}\left(\max_{1\leq k\leq n} \|X_k I\{u_n^p < \|X_k\|^p \leq n\} - \mathbb{E}XI\{u_n^p < \|X\|^p \leq n\}\| \right)^p \\
&\leq 2^{p+1}\mathbb{E}\|U_n^{(2)} - \mathbb{E}U_n^{(2)}\|^p,\ n \geq 1.
\end{aligned}$$

It thus follows from (3.35) and $\mathbb{E}\|X\|^p < \infty$ that

$$\begin{aligned}
&\sum_{n=1}^{\infty} \frac{\mathbb{E}\max_{1\leq k\leq n} \|X_k I\{u_n^p < \|X_k\|^p \leq n\}\|^p}{n^2} \\
&\leq 2^{2p} \sum_{n=1}^{\infty} \frac{\mathbb{E}\|U_n^{(2)} - \mathbb{E}U_n^{(2)}\|^p}{n^2} + 2^{p-1} \sum_{n=1}^{\infty} \frac{\mathbb{E}\|X\|^p}{n^2} \\
&< \infty,
\end{aligned}$$

and hence, by Lemma 5.4 of Li, Qi, and Rosalsky [10], noting that $\mathbb{P}\left(\|X\|^p > u_n^p\right) \leq n^{-1},\ n \geq 1$, we get that

$$\sum_{n=1}^{\infty} \frac{\mathbb{E}\|X\|^p I\left\{u_n^p < \|X\|^p \leq n\right\}}{n} < \infty. \tag{3.36}$$

Using partial integration, one can easily see that, for $n \geq 1$,

$$\left| \mathbb{E}\|X\|^p I\left\{u_n^p < \|X\|^p \leq n\right\} - \int_{u_n^p}^{n} \mathbb{P}\left(\|X\|^p > t\right) dt \right| \leq \frac{u_n^p}{n} + n\mathbb{P}\left(\|X\|^p > n\right). \tag{3.37}$$

Since $\mathbb{E}\|X\|^p < \infty$, we have

$$\sum_{n=1}^{\infty} \frac{n\mathbb{P}\left(\|X\|^p > n\right)}{n} = \sum_{n=1}^{n} \mathbb{P}\left(\|X\|^p > n\right) < \infty, \tag{3.38}$$

and, by Lemma 3.5, (3.14) holds. We thus see that (3.28) follows from (3.36), (3.37), (3.38), and (3.14) thereby completing the proof of the necessity half of Theorem 2.2. □

Proof of Theorem 2.3 We only need to consider the case where $q > 1$ since for the case where $q = 1$, Theorem 2.3 is Theorem 2.3 of Li, Qi, and Rosalsky [10]. Note that

$$\left\{\max_{1\leq k\leq n} \|X_k\| > n \text{ i.o.}(n)\right\} = \{\|X_n\| > n \text{ i.o.}(n)\}$$

and for $p = 1$,

$$U_n = \sum_{k=1}^{n} X_k I\left\{\|X_k\| \leq n\right\},\ n \geq 1.$$

By the Borel-Cantelli lemma, it thus follows from $\mathbb{E}\|X\| < \infty$ that

$$\mathbb{P}\left(\max_{1\leq k\leq n} \|X_k\| > n \text{ i.o.}(n)\right) = 0$$

and hence

$$\mathbb{P}\left(S_n - U_n \neq 0 \text{ i.o.}(n)\right) = 0,$$

which ensures that

$$\sum_{n=1}^{\infty} \frac{1}{n}\left(\frac{\|S_n - U_n\|}{n}\right)^q < \infty \text{ a.s.} \tag{3.39}$$

and by the Mourier [16] SLLN, it follows from (2.4) that

$$\begin{aligned} &\lim_{n\to\infty} \frac{U_n - n\mathbb{E}(XI\{\|X\| \leq n\})}{n} \\ &= \lim_{n\to\infty}\left(\frac{S_n}{n} - \mathbb{E}(XI\{\|X\| \leq n\})\right) - \lim_{n\to\infty}\frac{S_n - U_n}{n} \\ &= 0 \text{ a.s.} \end{aligned} \tag{3.40}$$

We now show that

$$\sum_{n=1}^{\infty} \frac{1}{n}\left(\frac{\|U_n - n\mathbb{E}(XI\{\|X\| \leq n\})\|}{n}\right)^q < \infty \text{ a.s.} \tag{3.41}$$

Since $\mathbf{B}$ is of stable type 1, the Maurey-Pisier [15] theorem asserts that it is also of stable type $1+\delta$ for some $0 < \delta < q-1$ and hence

$$\mathbb{E}\left\|U_n - n\mathbb{E}(XI\{\|X\| \leq n\})\right\|^{1+\delta} \leq C_{17}\mathbb{E}\left(\|X\|^{1+\delta} I\{\|X\| \leq n\}\right),\ n \geq 1.$$

Thus, by (3.15) (with $p = 1$) of Lemma 3.5, we conclude that

$$\sum_{n=1}^{\infty} \frac{1}{n}\mathbb{E}\left(\frac{\|U_n - n\mathbb{E}(XI\{\|X\| \leq n\})\|}{n}\right)^{1+\delta} < \infty$$

and hence

$$\sum_{n=1}^{\infty} \frac{1}{n} \left(\frac{\|U_n - n\mathbb{E}(XI\{\|X\| \le n\})\|}{n} \right)^{1+\delta} < \infty \text{ a.s.},$$

which, together with (3.40), ensures that (3.41) holds since $q > 1 + \delta$. Note that

$$S_n = (S_n - U_n) + (U_n - n\mathbb{E}(XI\{\|X\| \le n\})) + n\mathbb{E}(XI\{\|X\| \le n\}),\ n \ge 1.$$

We thus see that (2.3) (with $q > 1$) follows from (3.39), (3.41), and the second half of (2.4) (with $q > 1$).

Conversely, by Theorem 1.2 and the Mourier [16] SLLN, it follows from (2.3) that $\mathbb{E}X = 0$ and $\mathbb{E}\|X\| < \infty$ and hence (3.39) and (3.41) (since $\mathbf{B}$ is of stable type 1) hold. Note that

$$n\mathbb{E}(XI\{\|X\| \le n\}) = S_n - (S_n - U_n) - (U_n - n\mathbb{E}(XI\{\|X\| \le n\}))\,,\ n \ge 1.$$

It thus follows from (2.3), (3.39), and (3.41) that

$$\sum_{n=1}^{\infty} \frac{\|\mathbb{E}(XI\{\|X\| \le n\})\|^q}{n} = \sum_{n=1}^{\infty} \frac{1}{n} \left(\frac{\|n\mathbb{E}(XI\{\|X\| \le n\})\|}{n} \right)^q < \infty$$

and hence (2.4) holds (with $q > 1$). The proof of Theorem 2.3 is complete. □

4. Three Examples

Li, Qi, and Rosalsky [10] provided three examples (see, Examples 5.1, 5.2, and 5.3 of Li, Qi, and Rosalsky [10]) for illustrating the necessary and sufficient conditions that they obtained for (2.3) for the case where $q = 1$. In this section we provide three examples to illustrate our Theorems 1.2, 2.2, and 2.3.

Example 4.1. Let $1 < r < p < 2$ and let X be a real-valued symmetric random variable such that

$$\mathbb{P}(X = 0) = b \text{ and } \mathbb{P}(|X| > t) = \int_t^{\infty} \frac{1}{x^{p+1} \ln^r t} dt,\ t \ge e,$$

where $b = 1 - \int_e^{\infty} \frac{1}{x^{p+1} \ln^r x} dx$. Then $\mathbb{P}(|X| > t) \sim \frac{1}{pt^p \ln^r t}$ as $t \to \infty$ and hence, for $1 \le q < p$,

$$\mathbb{P}^{q/p}\left(|X|^q > t\right) = \mathbb{P}^{q/p}\left(|X| > t^{1/q}\right) \sim (q^r/p)^{q/p} t^{-1} (\ln t)^{-rq/p} \text{ as } t \to \infty.$$

We then see that

$$\int_0^{\infty} \mathbb{P}^{q/p}\left(|X|^q > t\right) dt \begin{cases} < \infty & \text{if } p/r < q < p, \\ \\ = \infty & \text{if } 1 \le q \le p/r. \end{cases}$$

It is also easy to check that $\mathbb{E}|X|^p \ln(1 + |X|) = \infty$ and $\mathbb{E}|X|^q = \infty$ for all $q > p$. By Theorem 2.2 and Remark 1.2, for this example, $X \in SLLN(p, q)$ if and only if $p/r < q < \infty$. However, by Corollary 2.2, (1.1) holds if and only if $p/r < q < p$. This means that, if (1.1) holds for some $q = q_1 > 0$, one cannot conclude that (1.1) holds for either $0 < q < q_1$ or $q > q_1$.

Example 4.2. Let $1 < p < 2$ and let X be a real-valued symmetric random variable with density function

$$f(x) = \frac{b}{|x|^{p+1} (\ln |x|)(\ln \ln |x|)^2} I\{|x| > 3\},$$

where $0 < b < \infty$ is such that $\int_{-\infty}^{\infty} f(x)dx = 1$. Clearly, we have that $\mathbb{E}X = 0$ and $\mathbb{E}|X|^p < \infty$. Since

$$\mathbb{P}(|X| > x) \sim \frac{2b/p}{x^p(\ln x)(\ln\ln x)^2} \text{ as } x \to \infty,$$

we see that

$$u_n \sim \frac{(2bn)^{1/p}}{(\ln n)^{1/p}(\ln\ln n)^{2/p}} \text{ as } n \to \infty$$

and hence, for all sufficiently large n,

$$\begin{aligned}
\int_{u_n^p}^{n} \mathbb{P}\left(|X|^p > t\right) dt &= \int_{u_n^p}^{n} \mathbb{P}\left(|X| > t^{1/p}\right) dt \\
&\geq \int_{\frac{3bn}{(\ln n)(\ln\ln n)^2}}^{n} \frac{b}{t(\ln t)(\ln\ln t)^2} dt \\
&\geq \frac{b}{(\ln n)(\ln\ln n)^2} \int_{\frac{3bn}{(\ln n)(\ln\ln n)^2}}^{n} \frac{1}{t} dt \\
&\sim \frac{b}{(\ln n)(\ln\ln n)} \text{ as } n \to \infty.
\end{aligned}$$

Note that

$$\sum_{n=3}^{\infty} \frac{b}{n(\ln n)(\ln\ln n)} = \infty$$

and so

$$\sum_{n=1}^{\infty} \frac{\int_{\min\{u_n^p, n\}}^{n} \mathbb{P}\left(|X|^p > t\right) dt}{n} = \infty.$$

By Theorem 2.2 and Remark 1.2, we thus conclude that $X \notin SLLN(p,q)$ for this example for all $0 < q \leq p$.

Let $1 < p < 2$ and let $\{X_n;\ n \geq 1\}$ be a sequence of independent copies of a symmetric real-valued random variable X. Then, by either Theorem 2.2 or Theorem 2.4, the following three statements are equivalent:

(i) $\mathbb{E}X = 0$ and $\mathbb{E}|X|^p < \infty$;
(ii) $X \in SLLN(p,q)$ for some $q > p$;
(iii) $X \in SLLN(p,q)$ for all $q > p$.

However, the following example shows that this is not true when $p = 1$.

Example 4.3. Let X be a real-valued random variable such that

$$\mathbb{P}\left(X = -\frac{1}{1-a}\right) = 1 - a \text{ and } \mathbb{P}(X > x) = \int_{x}^{\infty} \frac{1}{t^2(\ln t)(\ln\ln t)^2} dt,\ x \geq e^e$$

where $a = \int_{e}^{\infty} \frac{1}{t^2(\ln t)(\ln\ln t)^2} dt$. Then $\mathbb{E}X = 0$, $\mathbb{E}|X| < \infty$, and, for all sufficiently large n,

$$\mathbb{E}XI\{|X| \leq n\} = -\mathbb{E}XI\{|X| > n\} = -\int_{n}^{\infty} \frac{1}{t(\ln t)(\ln\ln t)^2} dt = -\frac{1}{\ln\ln n}.$$

Note that
$$\sum_{n=2}^{\infty} \frac{1}{n(\ln \ln n)^q} = \infty \text{ for all } q > 1.$$
Thus for this example, by either Theorem 2.3 or Theorem 2.4, $X \notin SLLN(1, q)$ for all $q > 1$.

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