Strengths and Drawbacks of Voting Methods for Political Elections

Eric Erdmann

University of Minnesota Duluth

Department of Mathematics and Statistics

Abstract

Our current method of electing government officials, **plurality voting**, has both drawbacks and strengths. One of the drawbacks is vote splitting which was identified as a problem as early as the 1912 United States presidential election. In this election a Democrat won because two Republican candidates split the Republican vote. A strength of plurality voting is its simplicity. In this report various drawbacks and strengths of voting methods will be discussed and analyzed, including the combination of criteria that Arrow's Impossibility Theorem states no rank-ordered voting method can satisfy. Several cities around the country have experimented recently with **instant runoff voting (IRV)**. Some of these cities have experienced dissatisfaction with this method and have eliminated it after only a few elections. Some reasons for this quick turnaround will be discussed in terms of the drawbacks of instant runoff voting. Another voting method used in political elections, the **Borda count**, will be discussed. Teaming is a serious drawback of this method. A basic version of **range voting** is presented in this report. It is related to product ratings which are widely used on the internet. This type of range voting appears to have less serious drawbacks than plurality, IRV and the Borda count, but it has not been used for political elections. Tied political elections in the United States have led to costly runoffs and ballot recounts. For instance, the 2008 United States Senate election in Minnesota cost the state over $12 million. In this report, the results of simulations that counted the frequency of tied elections for plurality, instant runoff voting and range voting are presented. It is found that among these three types of voting methods, tied elections occur most often for instant runoff voting and least often for range voting.

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Introduction

Voting method options have been debated for hundreds and maybe thousands of years. In France during the 1700's, the Marquis de Condorcet and Jean-Charles de Borda both strived to construct the perfect voting method. At the time, they were rivals, both having voting methods named after them. Both Borda and Condorcet tried to make each other's methods look bad. When Condorcet showed that Borda's method was susceptible to manipulation, Borda replied: "My scheme is intended only for honest men" (Poundstone, 2008). Both of these methods have not been widely used since the early 1800's; however they are still relevant to the conversation on voting methods today.

The debate over which voting method is the best and should be used continues. In the United States there are advocates for numerous voting methods. Instant runoff voting (IRV) is supported by a non-profit organization called FairVote. This organization often makes claims that are not backed up by facts. This includes claiming that IRV always produces a majority winner (Slatky, 2010). FairVote also downplays the negative aspects of the method. For example the problem of non-monotonicty is said to have little if any real world impact (FairVote, 2011). However, this problem makes IRV elections unpredictable since voting for one's favorite candidate can hurt the candidate's chances of winning, rather than helping them. According to Poundstone (2008) some people claim that "these mathematical 'paradoxes' that, while in theory are interesting for mathematicians to doodle around with on their sketch pads, in fact have no basis in reality". Mathematicians are concerned with this problem, but so should any person participating in an IRV election.

A mathematician named Donald Saari currently advocates the Borda count. He claims that this method is the only perfect method (Saari, 2001). At the same time, another mathematician, Warren Smith, claims that the Borda count is uniquely bad for use in political elections (Smith, 2011). Warren Smith prefers range voting and is not without his own over-the-top claims. Published on his website are claims that if the United States used range voting, 100,000,000 lives would be saved over the next 50 years (Smith, 2011).

In an investigation into the strengths and drawbacks of various voting methods, one must recognize that this maneuvering is going on among several groups of people who support competing voting methods. Therefore, when creating a report of this nature, one should be critical of all sources and try to eliminate unjustified claims, discern unbiased information and make reasonable inferences.

Section 1: Background Information

The simplest kind of an election is one in which there are only two candidates. In 1952, Kenneth May proved that majority rule is the best election method for two-candidate elections.

Definition 1.1	**Majority Rule**
For a two-candidate election, each voter indicates a preference for one of two candidates and the candidate with the most votes wins. For elections with three or more candidates, a candidate wins a majority rule election with first place support from voters or votes constituting more than half of the total number of votes in the election. (Garfunkel, 2006)	

Theorem 1.1 — **May's Theorem(1952)**

Among all two-candidate voting methods with an odd number of voters, majority rule is the only one that satisfies the following three conditions:

1. Treats all voters equally
2. Treats both candidates equally
3. If a new election were held and a single voter changes his or her vote from the loser of the previous election to the winner, with everyone else voting exactly as before, the outcome of the new election would be the same as the outcome of the previous election.

(Garfunkel, 2006)

Because of May's Theorem there is a wide consensus that for elections with only two candidates, majority rule is the best voting method. However, when there are more than two candidates running in an election, there is no voting method that is obviously the best. Another major theorem in voting theory, Arrow's Impossibility Theorem, highlights major problems associated with selecting an election winner when there are three or more candidates. Two concepts that Arrow's Impossibility Theorem utilizes are Independence of Irrelevant Alternatives and the Pareto condition. It also requires the use of preference lists which can also be in the form of a ranked order ballot. In an election, a preference list is given by each voter. The list contains an ordering of the candidates, with the voter's most preferred candidate listed first and least preferred candidate listed last. A voter is free to include as many or few of the candidates running in an election in the preference list.

Definition 1.2	**Independence of Irrelevant Alternatives (IIA)**
It is impossible for Candidate B to switch from losing to winning unless at least one voter reverses his or her preference ordering of Candidate B and the election winner. (Garfunkel, 2006)	

Definition 1.3	**Pareto Condition (Also Called Unanimity)**
If every voter prefers Candidate A to Candidate B, then Candidate B is not among the winners of the election. (Garfunkel, 2006)	

Theorem 1.2	**Arrow's Impossibility Theorem (1951)**
With three or more candidates and any number of voters, if a voting method always produces a winner, has strict transitive ballots (each voter supplies a strict ranking of the candidates), and satisfies the Pareto condition and IIA, then it is a dictatorship. (Saari, 2001)	

There are other versions of Arrow's Impossibility Theorem, some of which do not require strict transitive ballots (that is a voter can give two or more candidates equal ranking). For instance, see Geanakaplos (2005). Arrow's theorem is not obvious, and the proof is not provided in this report. To examine multiple proofs of Arrow's Impossibility Theorem see Geanakaplos (2005). A more accessible version of Arrow's Impossibility Theorem is stated in terms of the Condorcet Winner Criterion (CWC) and IIA. The CWC uses Condorcet's method.

Definition 1.4	**Condorcet's Method**
A candidate wins a Condorcet election if the candidate defeats every other candidate in a one-on-one contest using majority rule. (Garfunkel, 2006) Note that a Condorcet winner is unique.	

Definition 1.5	**Condorcet Winner Criterion (CWC)**
A voting method satisfies the Condorcet Winner Criterion (CWC) provided that, for every possible sequence of preference list ballots, either there is no Condorcet winner, or the voting method produces the same winner for this election as does Condorcet's method. (Garfunkel, 2006)	

In the following proof and throughout this report example elections are presented as follows. Voters' preference lists are summarized in tables. The election name is given at the top of the table. The voters' names or number of voters are listed in the left column of the table. The right column includes the position on the ballot each of the voters gave each of the candidates, with the candidate listed first being in first place. To clarify, a table with A>B>C listed under preference indicates that the voter(s) rank candidate A in first place (most preferred), B in second, and C in third (least preferred).

Theorem 1.3	Arrow's Impossibility Theorem (Weak Version)

With three or more candidates and an odd number of voters, there does not exist-and there never will exist-a rank-ordered social welfare method that satisfies both the Condorcet Winner Criterion (CWC) and Independence of Irrelevant Alternatives (IIA) and that always produces at least one winner in every election. (Garfunkel, 2006)

Proof

The following proof shows that if a "voting rule" were to be found to satisfy both CWC and IIA, then there would not be a winner of Election E at the right (called Condorcet's voting paradox).

Proof:

Suppose there exists a voting rule that satisfies both CWC and IIA.

Election E

Voter	Preference
Jo	A>B>C
Mo	B>C>A
Bo	C>A>B

Claim 1: A is a loser of E

C is the Condorcet winner of E1, thus by CWC, C is the unique winner of E1. Thus A is a loser of E1. Now by IIA, A remains a loser if Mo reverses his ranking of B and C. Therefore A is a loser of E.

Election E1

Voter	Preference
Jo	A>B>C
Mo	C>B>A
Bo	C>A>B

Claim 2: B is a loser of E

A is the Condorcet winner of E2, thus by CWC, A is the unique winner of E2. Thus B is a loser of E2. Now by IIA, B remains a loser if Bo reverses his ranking of A and C. Therefore B is a loser of E.

Election E2

Voter	Preference
Jo	A>B>C
Mo	B>C>A
Bo	A>C>B

Claim 3: C is a loser of E

B is the Condorcet winner of E3, thus by CWC, B is the unique winner of E3. Thus C is a loser of E3. Now by IIA, C remains a loser if Jo reverses his ranking of A and B. Therefore C is a loser of E.

Election E3

Voter	**Preference**
Jo	B>A>C
Mo	B>C>A
Bo	C>A>B

Therefore candidates A, B, and C are all losers of Election E. Thus any rank-ordered social welfare method that satisfies both IIA and CWC does not always produce a winner and therefore is not a valid voting method.

In this report, example elections are given illustrating how, more specifically, plurality, IRV and the Borda count violate the conditions of Arrow's Impossibility Theorem, namely CWC and IIA. In addition to CWC and IIA, other desirable properties of voting methods include majority rule (Definition 1.1), monotonicity, and clone independence. Monotonicity (see Definition 1.6 below) and clone independence (see Definitions 1.7.1-3 below) are properties that have to do with how the outcome of an election can change in response to changes in voters' ballots.

Definition 1.6	**Monotonicity**
Receiving more first place votes or a higher ranking can only benefit a candidate's chances of winning an election. Similarly, receiving fewer first place votes or a lesser ranking can only harm a candidate's chances of winning an election. (Garfunkel, 2006)	

Definition 1.7.1	Cloning: Vote Splitting
Vote splitting refers to the phenomenon in which the distribution of votes among similar candidates reduces the chance of winning for each of the similar candidates, and correspondingly increases the chance of winning for a dissimilar candidate. (Poundstone, 2008)	

Definition 1.7.2	Cloning: Vote Teaming
Vote teaming is an effect in which adding similar candidates to an election increases the chance of winning for each of the similar candidates. (Smith, 2011)	

Definition 1.7.3	Clone Independence
A voting method that does not suffer from either type of cloning, vote splitting or vote teaming, is said to be clone independent.	

This report contains information and examples from various sources. References are cited for work not original to this report. Many of the sources available present biased information promoting one or another voting method by only discussing and occasionally exaggerating the positive aspects of the method, while down-playing its drawbacks. Also, biased sources promoting a particular voting method may only present negative aspects or distortions of other voting methods. In this report there has been an attempt to avoid bias in order to investigate voting methods from an analytical and objective viewpoint. This report expands on standard mathematical analysis of voting methods by including recent real-world political election examples and examining practical issues associated with using the methods for political elections in the United States today.

Section 2: Plurality Voting

Plurality voting is the most widely used voting method for political elections in the United States. In plurality elections each voter places a vote for one candidate. The candidate who receives the most votes wins (Poundstone, 2008). Advantages of plurality include that it is easy for voters, as each voter only has to decide on a favorite candidate, and it is easy to tabulate and report results. Another advantage of plurality is that it satisfies majority rule. Unfortunately, plurality does not satisfy some other desirable criteria for a voting method. Examples 2.1 and 2.2 demonstrate how plurality does not satisfy either of the conditions of Arrow's Impossibility Theorem (Theorem 1.3), IIA and CWC.

Example 2.1	**Plurality Does Not Satisfy Independence of Irrelevant Alternatives (IIA)**

For plurality, only each voter's 1st place preference is considered. In Election 1, 10 people vote for A, 5 vote for B and 6 vote for C. Therefore A wins the plurality vote with a total of 10 votes. In Election 2, the 6 people who voted for C now change their preference ranking as shown and vote for B. Even though this change does not affect the relative rankings of A and B, it results in B winning Election 2 with a total of 11 votes. This demonstrates that plurality voting does not satisfy IIA.

Election 1

Number of Voters	**Preference**
10	A>C>B
5	B>C>A
6	C>B>A

Election 2

Number of Voters	**Preference**
10	A>C>B
5	B>C>A
6	B>C>A

Example 2.2	Plurality Does Not Satisfy the Condorcet Winner Criterion (CWC)

In this election, B wins the plurality election with a total of 10 votes. When the pairwise rankings are examined we find that voters prefer C to A by 16 to 5 and they prefer C to B by 11 to 10. Therefore C is the Condorcet winner. But since C is not also the plurality winner, this example demonstrates that plurality does not satisfy the CWC.

Number of Voters	Preference
5	A>C>B
10	B>C>A
6	C>A>B

Another problem with the plurality method is that it suffers from vote splitting. Also, it can elect a candidate approved of by only a minority of voters over another candidate approved of by a majority. Example 2.3 demonstrates these drawbacks.

Example 2.3	With Plurality, Vote Splitting Can Lead to Winners Without Majority Support

1912 United States Presidential Election

Candidate	Political Party	Percent of Vote
Taft	Republican	23.2%
Roosevelt	Republican	27.4%
Wilson	Democrat	41.8%

In the 1912 Presidential Election, Taft got the party nomination for the Republicans. Roosevelt, another Republican who ran, was well liked and had already been president for 8 years. Because of this, the 50.6% (a majority) of people voting Republican had to choose whether to vote for Roosevelt or Taft. Roosevelt ended up getting 27.4% of the vote, while Taft received 23.2%. Thus Taft and Roosevelt split the Republican vote and likely spoiled the election for each other. This enabled the Democrat Woodrow Wilson to win the election with 41.8% of the vote, which was a minority (Poundstone, 2008). It is conceivable that Wilson was only approved of by a minority of voters while at least one of the Republican candidates had majority approval.

Plurality does not take into account very much candidate preference information from voters, as only a voter's favorite candidate is counted. This limits the degree to which the will of the voters can be ascertained. This point is illustrated in Example 2.4. If the voting method would have taken into account the approval rating of each candidate, the outcome would have been different.

Example 2.4	**Plurality Voting does Not Use Much Information From Voters' Preference Lists**

2008 MN U.S. Senate Election				
Candidate	**Votes Before Recount**	**Votes After Recount**	**Approval**	**Disapproval**
Al Franken	1,211,375 (41.98%)	1,212,431 (41.991%)	47%	51%
Norm Coleman	1,211,590 (41.988%)	1,212,206 (41.984%)	51%	48%
Dean Barkley	437404 (15.158%)	437,505 (15.153%)	53%	33%

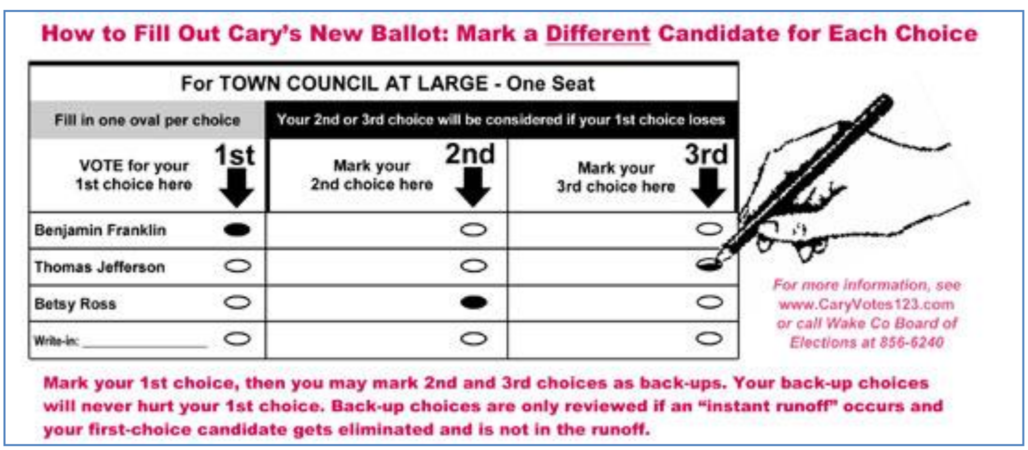
The winner of the election using plurality voting was Al Franken, However he had the lowest approval and highest disapproval rating among the candidates. (Wikipedia, 2011)

The other voting methods examined in this report allow voters to articulate their preferences for all of the candidates. An alternative voting method that eliminates or lessens the problems of plurality voting, while at the same time does not introduce new problems would clearly be an improvement. However, no method has been found to eliminate all of these problems. Instead it has been a challenge to compare various voting methods as each has a different combination of drawbacks that must be analyzed.

Section 3: Instant Runoff Voting

There is movement around the United States to change political elections from plurality voting to instant runoff voting (IRV). Cities that have recently adopted IRV include: San Francisco, Oakland, Minneapolis, and St. Paul. Others including Cary, North Carolina, Pierce County, Washington, Aspen Colorado, and Burlington Vermont have tried IRV and switched back to plurality voting (Gram, 2010). One of the most publicized examples occurred in Burlington, Vermont, in which a 2009 mayoral election displayed significant drawbacks of IRV. In March of 2010, the voters of Burlington voted against further use of IRV (Gierzynski, 2009). Another set-back for IRV happened in May of 2011 when more than two-thirds of British voters opposed replacing plurality to elect members of parliament with the alternative vote method, which is what Britain calls IRV (BBC News, May 2011). Locally, in the last few years some interest has been expressed concerning the possibility of changing the city of Duluth's municipal elections to IRV.

When using IRV for political elections each voter submits a ranked ordered ballot indicating preference ordering for candidates the voter deems acceptable. Ballots are similar to the following:



http://www.instantrunoffvoting.us/

The determination of a winner is carried out in rounds. In the first round each candidate receives one vote for each ballot on which he/she is marked as first choice. If a candidate secures a majority of votes cast, that candidate wins. If not, the candidate with the fewest votes is eliminated and a second round of counting takes place, with each ballot counted as one vote for the advancing candidate who is ranked highest on that ballot. This process continues until either a candidate has a majority of votes or two candidates remain. If there are two candidates remaining, the winner is the one with the most votes. (Poundstone, 2008)

Example 3.1	**IRV Election**

To the right is an example election. Included is the preference list for the voters, along with the round by round results of the runoff. In this election, Candidate C has the least first place votes, so Candidate C is eliminated after round 1. All voters who ranked C first on their ballot ranked A second; therefore, all of C's votes transfer to A in round 2. Now A has a majority, so A beats B with 5 votes to B's 4 votes.

Number of Voters	**Preference**
3	A>B>C
4	B>C>A
2	C>A>B

Candidate	**Round 1**	**Round 2**
A	3	5
B	4	4
C	2	-

One benefit IRV has over plurality is that voters are allowed to express a full preference list of all candidates rather than only their first place preference Also IRV is clone independent as illustrated in Example 3.2

Example 3.2	**IRV is Clone Independent**

1912 United States Presidential Election		
Candidate	**Round 1**	**Round 2**
Roosevelt	27.4%	50.6%
Taft	23.2%	-
Wilson	41.8%	41.8%

In the 1912 United States presidential election, two candidates that had spent time as president were both running as Republicans, Theodore Roosevelt and William Taft. They were challenging a Democrat, Woodrow Wilson, who ended up winning the election. The table shows how the election could have possibly resulted if IRV had been used. This is assuming the republican voters would rank both republican candidates above the democrat candidate (Poundstone, 2008).

Although IRV has some benefits over plurality, it has drawbacks as well. One of the benefits sometimes claimed by IRV supporters is that an IRV winner gets a majority of votes. However, in practice not all voters submit a complete preference list of all of the candidates, and so, while a winner receives a majority among voters whose ballots remain in the final round of the runoff, this may not be a majority among all who voted in the election. Example 3.3 demonstrates this phenomenon.

Example 3.3	If All Voters Do Not Rank All Candidates, the Winner of an IRV Election does Not Always Receive a Majority of Votes

In this election, voters only rank candidates they are willing to vote for. Here A wins with 4 of the 7 votes from the final round, which is only 4 out of 9 of the total votes. Note that losing Candidate B is supported by a majority and is preferred over Candidate A on 5 of the 9 ballots. So here we have an example where IRV elects a candidate with minority support instead of electing the candidate with majority support.

Number of Voters	Preference
4	A>B>C
2	B
1	C>D>B
2	D>C>B

Candidate	Round 1	Round 2	Round 3
A	4	4	4
B	2	2	-
C	1	-	-
D	2	3	3

The failure of IRV to elect a winner with a majority of the vote can also be seen in real-world elections. Similarly to plurality voting, there are IRV elections in which no candidate receives a majority of the vote such as Oakland, California's mayoral election held in 2010.

Example 3.4	**The Winner of the Oakland Mayoral IRV Election of 2010 did Receive a Majority of the Votes**

Candidate	**Round 1**		**Round 9**	**Round 10**
Don Perata	40342 (33.73%)		45465 (40.16%)	51872 (49.04%)
Jean Quan	29266 (24.47%)		35033 (30.94%)	53897 (50.96%)
Rebecca Kaplan	25813 (21.58%)		32719 (28.90%)	-
Exhausted Ballots	0		6284	13667
Total	122268		122268	122268

In this election, Ranked Choice Voting reported that Jean Quan won with a majority (50.96%). However, when comparing her total, 53897, with the total number of people who voted in the mayoral election, 122268, she received only 44.08% of the voter.

Note: Including write-ins, there were 11 candidates in this election. The table is only meant to summarize the final results so the other 8 candidates were left off. Also not shown are under-votes and over-votes. The full results can be seen here: http://www.acgov.org/rov/rcv/results/rcvresults_2984.htm

IRV also does not satisfy Independence of Irrelevant Alternatives (IIA). Example 3.5 illustrates this problem.

Example 3.5	IRV Does Not Satisfy Independence of Irrelevant Alternatives (IIA)

In Election 1, B is eliminated first and A wins the election. Now in Election 2, if the 2 people who voted C>B>A change their relative ranking of B and C, then A is eliminated first and B wins. Between these elections the relative rankings of A and B did not change, however the final ranking of A and B did change. Thus IRV does not satisfy the IIA criterion.

Election 1

Number of Voters	**Preference**
2	A>B>C
1	B>A>C
2	C>B>A

Election 2

Number of Voters	**Preference**
2	A>B>C
1	B>A>C
2	B>C>A

IRV also does not satisfy the Condorcet Winner Criterion (CWC). This can be seen again with a simple example.

Example 3.6	IRV Does Not Satisfy the Condorcet Winner Criterion (CWC)

If the pairwise rankings of the election to the right are examined, one finds that B beats both A and C, so B is the Condorcet Winner. However, A and not B wins the IRV election. Thus IRV does not satisfy CWC.

Number of Voters	**Preference**
2	A>B>C
1	B>A>C
2	C>B>A

Another desirable property which IRV does not satisfy is monotonicity.

Example 3.7	IRV does not Satisfy the Monotonicity Criterion

In Election 1, A is the IRV winner with 65 votes. Now for Election 2, suppose that 10 of the people who voted BCA in Election 1 decided to vote ABC, thus giving A more votes in the first round. When the instant runoff is carried out for Election 2, B is eliminated first rather than C, and C ends up winning with 51 votes. Thus giving A more first place votes has caused A to lose the election. Therefore by definition, IRV does not satisfy the monotonicity criterion.

Election 1

Number of Voters	Preference
39	A>B>C
35	B>C>A
26	C>A>B

Election 2

Number of Voters	Preference
49	A>B>C
25	B>C>A
26	C>A>B

Example 3.7 also highlights a possible relationship between monotonicity and the IIA criterion. Between Election 1 and 2 none of the voters change their relative ranking of A and C, however the overall ranking of candidates A and C changes. Candidate C is now the winner rather than Candidate A. Because of this, it seems that if an election method does not meet the monotonicity criterion, it certainly does not meet the IIA criterion. This however needs to be proved. Keep in mind that the converse is not true; this was seen with plurality elections. Plurality does not meet the IIA criterion, however it is monotonic. Therefore, when considering election methods, methods which are not monotonic are less desirable than methods that do not meet the IIA criterion.

Another problem that can be seen with IRV is higher probability for ties in elections as compared to plurality voting. This is because a tie is possible in each round of an IRV runoff. With this can come an increase in the probability of recounts and litigation. This problem will be investigated in Section 6. This can also lead to some chaotic behavior because the order in which candidates are eliminated can determine the IRV winner. The following is an example IRV election with a tie in two different rounds. Depending on who wins the tie breakers, there are three different possible winners.

Example 3.8	**Ties Among candidates to be Eliminated can cause Chaotic Behavior in IRV Elections**

With this election, there is a tie in the first round between C and D; as well as in the second round between B and C or D. The winners of these ties are selected at random. The winner of the election depends on who wins the tie breakers. If C is the random winner of the round one tie, and B is the winner of the round two tie, B wins the election. If C is the random winner of the round one tie, and C is the winner of the round two tie, A wins the election. On the other hand, if D is the random winner of the round one tie, and B is the winner of the round 2 tie, B wins the election. If D is the random winner of the round one tie, and D is the winner of the round 2 tie, D wins the election. Thus there are three possible different winners depending on who wins the random tie breaker.

Number of Voters	**Preference**
5	A>C>D>B
6	A>B>C>D
1	B>A>C>D
5	B>C>D>A
3	B>D>A>C
1	C>A>D>B
1	C>B>A>D
4	C>D>B>A
1	D>A>B>C
1	D>B>A>C
1	D>C>A>B
3	D>C>B>A

C wins the Round 1 coin toss and B the Round 2 coin toss:

Candidate	**Round 1**	**Round 2**	**Round 3**
A	11	12	14
B	9	10	18
C	6	10	
D	6		

C wins the Round 1 coin toss and C the Round 2 coin toss:

Candidate	**Round 1**	**Round 2**	**Round 3**
A	11	12	17
B	9	10	
C	6	10	15
D	6		

D wins the Round 1 coin toss and B the Round 2 coin toss:

Candidate	**Round 1**	**Round 2**	**Round 3**
A	11	12	14
B	9	10	18
C	6		
D	6	10	

D wins the Round 1 coin toss and D the Round 2 coin toss:

Candidate	**Round 1**	**Round 2**	**Round 3**
A	11	12	14
B	9	10	
C	6		
D	6	10	18

Another drawback of using IRV for political elections involving large numbers of voters is the amount of information and the centralization of the information required for carrying out the voting algorithm. Each of the other voting methods discussed in this paper requires a single vote total or score total to be sent from each polling place to a central location. However, when using IRV, this process of getting votes from each polling place to a central location may be problematic since a runoff cannot begin until all information is centralized. A summary of the ballots (preference lists) of all of the voters from every polling place needs to be centralized before the election algorithm can be run. Thus, much more information than for plurality voting needs to be sent and processed and processing errors and fraud may be harder to detect and trace.

A mayoral election held in Burlington, Vermont in 2009 displayed IRV's failure to satisfy CWC and non-monotonicity. Analysis of the published election summary (Gierzynsk,2009) revealed that one of the losing candidates was the Condorcet winner and had the IRV winner received about 750 additional first place votes he would have lost the election. Example 3.9 gives the round-by-round results of this election. The table of pairwise rankings shows that Montroll would be the winner of the election if Condorcet's method were used, as he beats both Kiss and Wright in pairwise contests. Specifically, Montroll beats Kiss 4067 to 3477 and he beats Wright 4567 to 3668. Thus, IRV does not satisfy CWC in this election. In the Example 3.9 table summarizing voters' preferences among the top three candidates, Kiss Wright and Montroll, notice that if 495 of the W>K>M voters and 258 of the W voters decided to vote for K, then W would have been eliminated first instead of M, and M would have beaten K in the final round . Knowing that more first place votes for K would have caused him to lose must have been unsettling for those who campaigned for him. Should they be praised for not successfully recruiting more support for him?

Example 3.9	The Burlington, Vermont 2009 Mayoral Election Displayed Many of the Possible Drawbacks of IRV

Actual Runoff Results:

CANDIDATE	Round 1	Round 2	Round 3
Kiss	2585	2981	4313
Wright	2951	3294	4061
Montroll	2063	2554	
Smith	1306		
Simpson	35		
Writ-ins	36		

Pairwise Rankings:

Candidate	Kiss	Montroll	Wright
Kiss		3477	4314
Montroll	4067		4567
Wright	4064	3668	

Possibility for Non-Monotonicity:

CANDIDATE	Round 1	Round 2	Round 3
Kiss	3338	2981	3755
Wright	2198	2541	
Montroll	2063	2554	4057
Smith	1306		
Simpson	35		
Writ-ins	36		

Voter Preference List:

Number of Voters	Preference
1332	M>K>W
767	M>W>K
455	M
2043	K>W>M
371	K>W>M
568	K
1513	W>M>K
495	W>K>M
1289	W

After the 2009 mayoral election in Burlington, the city decided to do away with IRV (Gram, 2010). Other places around the country in which IRV was recently instated have also reverted back to their previous way of electing officials. Cary, North Carolina used IRV in 2007 and repealed it shortly after because “ballots were mis-sorted, simple calculator mistakes were made and a non-public recount turned up missing votes. The winner did not receive the 50 percent plus one vote majority advocates claimed IRV would ensure in a single election” (Teslesca, 2009). Pierce County, Washington, which includes the city of Tacoma, used IRV in 2008. Afterwards, 63% of voters voted to eliminate IRV in 2009. In Pierce County, “Opponents say ranked-choice voting is confusing and costly. By the end of this year Pierce County will have spent $2.3 million on the voting system. The county has budgeted another $500,000 for next year” (Wickert, 2009). Finally, Aspen, Colorado used IRV in 2009, and 65% voted to eliminate IRV in 2010 “The ballots were “spit out of a black box,” as Councilman Dwayne Romero put it” (Wackerle,2010).

Although IRV corrects many of the problems with plurality, it has many of its own problems. IRV does not satisfy many of the desirable criteria of voting methods, including IIA, CWC, and montonicity.

Finally, when using IRV in an election, a voter submits a list of preferences without knowing how much of it will actually be used. As an example, consider a voter whose first choice is in the final round of the runoff but loses. This voter’s ranking of lower candidates will never be examined. This leads to the question, why not employ a voting method that uses all information submitted by the voters?

Section 4: The Borda Count

The Borda count is another voting method that uses a preference list. In an election with N candidates where N>2 (If N=2, it is a majority rules election), each voter will rank any or all of the N candidates from most preferred to least preferred. When used for political elections, voters should be instructed to only rank candidates whom they find acceptable. This prevents voters from helping to elect candidates they deem unacceptable. The ballots look similar to those for IRV. However, when tallying the ballots for the Borda count the voter's first preference is given a weight of N-1, the second a weight of N-2, the third N-3, and so on, with all unranked candidates receiving a weight of 0. If a voter chooses to rank all the candidates, the least preferred candidate gets a weight of 0. After the ballots are tallied, the candidate with the overall highest weighted sum becomes the winner of the election (Garfunkel, 2006).

Example 4.1	**An Election Using the Borda Count**

Number of Voters	**Preference**	**A**	**B**	**C**	**D**
150	A>C>D>B	450	0	300	150
125	B>D>C>A	0	375	125	250
40	D>C>A>B	40	0	80	120
	Total	**490**	**375**	**505**	**520**

<u>Calculating Total Points</u>

A=3(150)+1(40)= 490

B=3(125)= 375

C=2(40)+2(150)+1(125)= 505

D=3(40)+2(125)+1(150)=520

In this election Candidate A has 150 1st place votes which give A 450 points. Candidate A also has 40 third place votes which give A 40 points. Therefore A has a total of 490 points. Candidates B, C, and D have point totals that are calculated similarly. D is the winner of the election with 520 points, since this is the highest weighted sum.

The Borda count is currently used in many contexts. It is used in political elections in Kiribati and Nauru which are Micronesian nations, as well as in Slovenia. It is used differently in each of these countries. In Slovenia it is used to elect two of the ninety members of the National Assembly. In Kiribati it is used to select a field of presidential candidates for the election. Once the field is decided by the Borda count, plurality is used to select the winner of the candidates. In Nauru, a modification of the Borda count is used to select the parliament. Other uses of the Borda count include applications in sports, educational institutions, and professional societies. (Wikipedia, 2010)

The Borda count violates the IIA criterion of Arrow's Impossibility Theorem, as seen in Example 4.2.

Example 4.2	**The Borda Count Does Not Satisfy Independence of Irrelevant Alternatives (IIA)**

In Election 1, 5 voters submit the ranking A>C>B giving A 10 point, B 0 points, and C 5 points. Also 6 voters submit the ranking B>A>C giving A 6 point, B 12 points, and C 0 points. A wins with 16 points. Election 2 differs from Election 1 in that the 5 people who voted A>C>B now vote A>B>C. This change does not affect the relative ranking of A and B on any of the ballots. However, B ends up winning the election with 17 points, thus changing the overall ranking of A and B. Therefore the Borda count does not satisfy the IIA criterion.

Election 1

Number of Voters	**Preference**	**A**	**B**	**C**
5	A>C>B	10	0	5
6	B>A>C	6	12	0
	Total	**16**	**12**	**5**

Election 2

Number of Voters	**Preference**	**A**	**B**	**C**
5	A>B>C	10	5	0
6	B>A>C	6	12	0
	Total	**16**	**17**	**0**

Donald Saari is a Professor of Mathematics and Economics at the University of California Irvine who has researched voting methods. He advocates the Borda count. He modified Arrow's Impossibility Theorem (Theorem 1.3) to obtain a positive result concerning the Borda count. He replaces IIA, which he calls Binary Independence with a property he calls Intensity of Binary Independence. In this report it is called Intensity of Independence of Irrelevant Alternatives (IIIA). Saari's ballot-level intensity factor of the pairwise rankings of A and B is the number of candidates who appear between A and B in a voter's preference list. For example, if a voter's ballot has the preference: A>B>C, the intensity factors are expressed as: (A>B,0),(B>C,0),(A>C,1).

Definition 4.1	**Intensity of Independence of Irrelevant Alternatives (IIIA)**
It is impossible for Candidate B to switch from losing to winning unless at least one voter changes his or her intensity of preference between Candidate B and the election winner. (Saari, 2001)	

If a voter were to change his preference list A > B > C to A > C > B then, even though the relative ranking of A and B do not change, the intensity factor of the rankings of A and B has changed from 0 to 1. The changed intensity factors are (A>B,1),(C>B,0),(A>C,0). If this change occurred in an election and it affected the outcome between A and B, this would not violate IIIA because the voter's intensity of the preference for A over B changed. Donald Saari (2001) showed that if IIA is replaced with IIIA in Arrow's Impossibility Theorem (Theorem 1.2), the Borda count is the unique method that satisfies this modified theorem. For further justification, see pages 190-192 of Saari (2001).

Example 4.3	**The Borda Count Satisfies Intensity of Independence of Irrelevant Alternatives (IIIA)**

In Example 4.2 consideration of Election 1 and Election 2 shows that the Borda count does not satisfy IIA. However, between the two elections, the intensity factor of the A>B ranking changes, thus IIIA is not violated in this example .For further justification for why the Borda count satisfies IIIA, see pages 190-192 of Saari (2001).

Election 1

Number of Voters	**Ranking**	**A**	**B**	**C**	**AB Ranking**
5	A>C>B	10	0	5	(A>B,1)
6	B>A>C	6	12	0	(B>A,0)
	Total	**16**	**12**	**5**	

Election 2

Number of Voters	**Ranking**	**A**	**B**	**C**	**AB Ranking**
5	A>B>C	10	5	0	(A>B,0)
6	B>A>C	6	12	0	(B>A,0)
	Total	**16**	**17**	**0**	

The Borda count satisfies the monotonicity criterion since if a candidate's ranking by a voter is increased it will give the candidate more points and as a result cannot negatively affect the candidate's overall ranking. Another positive feature of the Borda count is that the full preference list of each voter is used. Recall that with IRV, in contrast, voters' lower preferences are only considered if their higher ranked candidates are eliminated. Thus with IRV, the full preference list of one voter may be utilized, while only the top preference of another voter is utilized.

Properties not satisfied by the Borda count include majority rule (Example 4.4), the Condorcet winner criterion (Example 4.5), and clone independence (Example 4.6).

Example 4.4	The Borda Count doe Not Satisfy Majority Rule

In this election, candidate B is the most preferred by a majority of the 11 voters, however candidate A is the Borda count winner. Therefore, the Borda count does not satisfy the majority rule criterion.

Number of Voters	Ranking	A	B	C
5	A>C>B	10	0	5
6	B>A>C	6	12	0
	Total	**16**	**12**	**0**

Example 4.5	The Borda Count Does Not Satisfy the Condorcet Winner Criterion (CWC)

Pairwise comparisons in this election show that B is favored over A by 6 of the 11 voters and B is favored over C by 6 of the 11 voters. Therefore B is the Condorcet winner. However B loses the Borda count election to A. Therefore, the Borda count does not satisfy the Condorcet winner criterion.

Number of Voters	Ranking	A	B	C
5	A>C>B	10	0	5
6	B>A>C	6	12	0
	Total	**16**	**12**	**0**

Example 4.6 | **The Borda Count Suffers from a Type of Cloning Called Teaming**

The bottom row of the table for Election 1 indicates that Candidate C wins this election with a total score of 355. In Election 2 a clone of A, say A’, enters the election. Assume that the voters who like A, like A’ almost as well, and the voters that dislike A, dislike A’ almost as much. Because A’ entered the election, the score for each voters first place candidate is 3 rather than 2, therefore having A’ in the election helps candidate A and causes him to win. Therefore the Borda count does not satisfy the clone independence criterion.

Election 1

Number of Voters	**Preference**	**A**	**B**	**C**
150	A>C>B	300	0	150
125	B>C>A	0	250	125
40	C>A>B	40	0	80
	Total	**340**	**250**	**355**

Election 2

Number of Voters	**Preference**	**A**	**A’**	**B**	**C**
150	A>A’>C>B	450	300	0	150
125	B>C>A’>A	0	125	375	250
40	C>A>A’>B	80	40	0	120
	Total	**530**	**465**	**375**	**520**

Neither IRV nor the Borda count appears to be a definite improvement over plurality for use in political elections in the United States today. This is because of the nonmonotonicity problem of IRV and the teaming problem of the Borda count. This motivates exploration of other potential substitutes for plurality.

Section 5: Range Voting

"According to Arrow's definition, non-ranked choice methods are not voting methods at all. The true lesson of Arrow's theorem is (more than anything else, in my opinion) that you should stay away from voting methods based on rank-order ballots" (Warren Smith, 2011). Indeed the proof of Arrow's Impossibility Theorem (weak version) assumes that a voting method requires ballots that have candidates ranked in order of preference. If a voter feels equally about two or more candidates, ranking the candidates will force the voter to distort his true preference by ranking one candidate above the other. In contrast, range voting ballots allows voters to more fully show their strengths of preferences among the candidates. With range voting, the voters give a numerical score (within a predetermined range) to each of the candidates. If a voter gives Candidate A a high score and Candidate B a low score, he is indicating a strong relative preference for A over B. When ballots are tallied the candidate with the highest summed score is the winner (Poundstone, 2008). When the range is 0 to 1, the method is called Approval Voting. In this report, the range will usually be 0 to 4 or 0 to 10. Example 5.1 displays what a ballot may look like.

Example 5.1 **Range Voting Ballot**

This is an example of a range voting ballot for a single voter. This voter's ballot indicates that he considers Adrian, Carl and Debbie to be acceptable candidates, with a greater preference for Carl and Debbie. In this way, a voter's intended preferences are more fully expressed than in a preference list.

Candidate	Score				
	0	**1**	**2**	**3**	**4**
Adrian		X			
Betty	X				
Carl					X
Debbie					X

Range Voting escapes the conclusions of Arrow's Impossibility Theorem because it is not a ranked choice voting method. However, let's consider the criteria of Arrow's Impossibility Theorem and how they might be adapted to fit range voting. There is debate over whether or not range voting satisfies the intent of IIA. It depends on how you interpret IIA for range voting, as IIA is defined only for rank order voting methods.

Example 5.2	**Range Voting Does Not Satisfy the Intent of Independence of Irrelevant Alternatives, First Interpretation.**

From Election 1, relative rankings can be gathered for each voter, Voter 1: A>C>B, Voter 2: B>A>C, and Voter 3: C>A>B. A wins Election 1.

In Election 2, Voter 1's relative rankings of B and C are changed but the A>B ranking is not changed. B wins Election 2, thus showing that range voting does not satisfy the intent of IIA.

Election 1

Candidate	**Voter 1 Scores**	**Voter 2 Scores**	**Voter 3 Scores**	**Total Scores**
A	10	6	3	**19**
B	6	10	1	**17**
C	7	2	8	**17**

Election 2

Candidate	**Voter 1 Scores**	**Voter 2 Scores**	**Voter 3 Scores**	**Total Scores**
A	10	6	3	**19**
B	9	10	1	**20**
C	7	2	8	**17**

The previous example shows that range voting does not satisfy IIA, if we determine candidate rankings from the ordering of scores on range-voting ballots. However this interpretation of IIA does not take into account that range voting allows for changes in scores that do not necessarily affect the ranking of the candidates. The IIA criterion can be examined slightly differently for range voting.

Rather than switching the relative ranking of candidates, a candidate that is considered the irrelevant alternative can be introduced. Remember that IIA is defined as: It is impossible for candidate B to switch from losing to winning unless at least one voter reverses the ordering of candidate B and the election winner on his ballot. Therefore adding a new candidate should not affect the final ranking order of the original candidates in the election if the method satisfies IIA. First the IIA criterion for range voting will be considered with scores remaining constant from the first election to the second election. This is called IIA-S (Cary, 2006). In example 5.3, notice that if none of the voters change their scores(the scores remain constant) for candidates A and B when a third candidate, C, joins the election, the overall ranking of candidates A and B will also not change.

Example 5.3	**Range Voting Does Satisfy Independence of Irrelevant Alternatives with Scores Remaining Constant (IIA-S)**

Election 1 has candidates A and B and 3 voters.

A wins the range voting election with score of 18 versus 17 for B.

A third candidate, C, runs in Election 2, and the voters give him scores as well. Assume that the scores for A and B are the same as they were in Election 1. Thus C in no way affects the result between A and B. Therefore, range voting satisfies IIA-S.

Election 1

Candidate	**Voter 1 Scores**	**Voter 2 Scores**	**Voter 3 Scores**	**Total Scores**
A	2	10	6	**18**
B	0	7	10	**17**

Election 2

Candidate	**Voter 1 Scores**	**Voter 2 Scores**	**Voter 3 Scores**	**Total Scores**
A	2	10	6	**18**
B	0	7	10	**17**
C	10	0	3	**13**

IIA-S essentially only considers how a voting method tabulates votes. It ignores any consideration of how voters' relative preferences for candidates change with the introduction of a new candidate. An alternative to IIA-S involves holding voters' sincere opinions of candidates, rather than their ballot-expressed preferences, constant. Call that IIA-O. This preference strength (sincere opinions voters have of candidates) does not change based on how many candidates are running. However the way each voter marks his ballot may change. It is assumed that each voter gives a most preferred candidate a 10 and least preferred candidate a 0. If a voter does not do this then he is not promoting his most preferred candidate over his least preferred to the full extent that he can expect other voters to. When there are only two candidates, these are the only possible scores. However, when a third candidate is added, one of the candidates might receive a score other than 0 or 10.

Example 5.4	**Range Voting Does Not Satisfy Independence of Irrelevant Alternatives with Sincere Opinions Remaining Constant (IIA-O)**

Election 1

Voters' Sincere Opinions :

Voter 1: A= 4, B= 10

Voter 2: A=10, B=8

Voter 3: A= 10, B= 7

Election Results:

Candidate	**Voter 1 Scores**	**Voter 2 Scores**	**Voter 3 Scores**	**Total Scores**
A	0	10	10	**20**
B	10	0	0	**10**

Election 1 has candidates A and B. Each of the three voters has an opinion about the candidates; this is then transferred into a range vote. Voter 1 prefers B over A so gives B a score of 10 and A a score of 0 (he gives full support to his favorite). The other voters do this as well to produce a result of A winning over B 20 to 10.

Election2

Voters' Sincere Opinions :

Voter 1: A= 4, B= 10, C= 0

Voter 2: A= 10, B= 8, C= 0

Election Results:

Candidate	**Voter 1 Scores**	**Voter 2 Scores**	**Voter 3 Scores**	**Total Scores**

Voter 3: A= 10, B= 7, C= 0		A	4	10	10	**24**
		B	10	8	7	**25**
		C	0	0	0	**0**

In Election 2, a third candidate, C, is introduced. None of the voters approve of C. The voters still feel the same about the other candidates, but now have to consider how they feel about C when assigning scores to each candidate. On the left are each voter's sincere opinions. When voter 1 votes, he considers how he feels about Candidate C when casting his ballot. He likes Candidate A better than Candidate C and expresses it with his vote, giving A a 4 instead of a 0 as in Election 1. These additional points for A do not change the relative ranking of candidates A and B on Voter 1's ballot. The scores that Voters 2 and 3 assign to their second preference also change. The result is that B wins Election 2. Adding an irrelevant alternative, C, who gets no points from any of the voters, changes the winner of the election from A to B. Therefore range voting does not satisfy IIA-O.

This example shows that range voting does not satisfy all interpretations of IIA. However range voting does supply more information than is considered by the original IIA criterion (Definition 2.1).

One could consider intensity when evaluating the IIA criterion for range voting much like Donald Saari did for the Borda Count. To examine this, an intensity factor similar to the one used in the Borda count in Section 4 will be defined. This will be called intensity of preference and denote it by (A>B: x), which can be read, A beats B by x points. For example if a voter gives Candidate A 10 points and Candidate B 4 points, this would be written: (A>B: 6). This will be used for examining the IIIA criteria (Definition 4.1).

Conjecture 5.1	**Range Voting Satisfies Intensity of Independent of Irrelevant Alternatives (IIIA)**
When using range voting it is impossible for Candidate B to switch from losing to winning unless at least one voter changes his or her intensity of preference for Candidate B and the election winner.	

Intuitively, range voting should satisfy the IIIA criterion. Therefore, if a third candidate is added, and affects how the voters score the candidates, it is expected that the outcome might change. Let's look at an example of this.

Example 5.5	**A Range Voting Election that Does Satisfy IIA Does Satisfy Intensity of Independence of Irrelevant Alternatives (IIIA)**

Election 1 has candidates A and B and three voters. The A>B intensity of preference (A>B:x) is also given. Candidate A wins Election 1, 20 to 10. In Election 2, Candidate C is introduced and B wins the election. Candidate C causes the A>B intensity of preference to change, and thus IIIA is not violated.

Election 1

Candidate	**Voter 1 Scores**	**Voter 2 Scores**	**Voter 3 Scores**	**Total Scores**
A	10	10	0	**20**
B	0	0	10	**10**
Intensity of AB Ranking	A>B: 10	A>B:10	B>A:10	

Election 2

Candidate	**Voter 1 Scores**	**Voter 2 Scores**	**Voter 3 Scores**	**Total Scores**
A	10	10	0	**20**
B	8	7	10	**25**
C	0	0	4	**4**
Intensity of AB Ranking	A>B:2	A>B:3	B>A:10	

The other criterion of Arrow's Impossibility Theorem to look at is the CWC. Example 5.6 shows that range voting does not satisfy CWC.

Example 5.6	**Range Voting Does Not Satisfy the Condorcet Winner Criterion (CWC)**

Candidate	**Voter 1 Scores**	**Voter 2 Scores**	**Voter 3 Scores**	**Total Scores**
A	2	10	3	**15**
B	0	7	10	**17**
C	10	0	2	**12**

In this election, Candidate A is the Condorcet winner; however Candidate B gets the most points and is therefore the winner of the election. Therefore range voting does not satisfy CWC.

Note that Example 5.6 shows that range voting, with the range 0–10, violates CWC. This shows that range voting in general (i.e., where the range is unspecified), violates CWC. It does not deny, however, that there might be certain ranges for which CWC is satisfied. Indeed approval voting, which is range voting with the range 0-1, does satisfy CWC. (Brams, 2008)

Since with range voting voters give the intensity of preference for the candidates, why not use this additional information when looking at voting method criteria? Could it be possible to modify the Condorcet Winner Criterion to be more fitting for voting methods that allow voters to express intensity of preference? This could be done similarly to the way the criterion for IIA was modified to come up with IIIA. This modification is called Intensity of Condorcet Winner Criteria (ICWC).

Definition 5.1	**Intensity of Condorcet Winner Criterion (ICWC)**

A voting method satisfies the Intensity of Condorcet Winner Criterion (ICWC) provided that, when all voters' intensity of preference between any pair of candidates is summed, the winner of the election is the candidate with the highest sum if intensity of preferences. (For example if (A>B:x) was the intensity of preference for a voter, x points would get added to the sum for candidate A)

Conjecture 5.2	**Range Voting Satisfies Intensity of Condorcet Winner Criterion (ICWC)**

When using range voting, if a candidate defeats every other candidate in a one-on-one contest based on the intensity of preference, then that candidate is the winner of the election.

Consider the election in Example 5.6 once again.

Example 5.7	**A Range Voting Election that Did Not Satisfy the CWC Does Satisfy the Intensity of Condorcet Winner Criterion (ICWC)**

Candidate	**Voter 1 Scores**	**Voter 2 Scores**	**Voter 3 Scores**	**Total Scores**
A	2	10	3	**15**
B	0	7	10	**17**
C	10	0	2	**12**

	Preference Strength				
Preference	**Voter 1 Scores**	**Voter 2 Scores**	**Voter 3 Scores**	**Total Scores**	**Overall Candidate Ranking**
A>B	2	3	0	5	**B>A:2**
B>A	0	0	7	7	
A>C	0	10	1	11	**A>C:3**

	C>A	8	0	0	8		
	B>C	0	7	8	15	**B>C:5**	
	C>B	10	0	0	10		

Let us look at the pairwise scores for candidates A and B alone. For Voter 1, A beats B by 2 points. For Voter 2, A beats B by 3 points. And for Voter 3, B beats A by 7 points. Therefore Candidate B beats A by 7 points and A beats B by a total of 5 points, therefore the overall preference strength of Candidates A and B is B>A:2, thus B is the pairwise winner. If scores for A and C are compared, A beats C by 10+1=11 points and C beats A by 8 points giving a preference strength of A>C:3, thus A is the pairwise winner. Now let's look at B and C, B beats C by a 7+8=15 points and C beats B by 10 points giving a preference strength of B>C:5, thus B is the pairwise winner. Therefore in a pairwise election, B beats both A and C and wins the election. Thus, this set of elections does not violate the Intensity of Condorcet Winner Criterion (ICWC).

Proofs of whether range voting satisfies IIIA and ICWC are still needed. The examples in this section show that at the very least, in some sets of elections where IIA and CWC are violated, the modified versions of IIA and CWC, IIIA and ICWC, are not violated.

Some other criteria that will be looked at as they pertain to range voting are majority rule, monotonicity, clone independence, and chance of ties. It is difficult to discuss the majority rule criterion as it pertains to range voting because more than one candidate could receive a majority of the support. This is because voters can award the same score to candidates. To analyze majority rule, assume that the candidate or candidates whom each voter gives the highest score to would be the one each voter would vote for in a majority rule election. In Example 5.8 notice that range voting does not satisfy the majority rule criterion.

Example 5.8	**The Winner of a Range Voting Election is Not Always the Winner of a Majority Rule Election**

Candidate	**Voter 1 Scores**	**Voter 2 Scores**	**Voter 3 Scores**	**Total Scores**
A	10	10	0	**20**
B	6	5	10	**21**
C	0	0	8	**8**

In this election, Candidate A is the majority rule winner, as he his preferred by two of the three candidates; however Candidate B gets the most points and is therefore the winner of the election. Therefore by the definition, range voting does not satisfy the majority rule criterion.

Despite failing the majority rule criterion, range voting will elect a candidate with the broadest support in terms of total points. Also if a majority of voters support a candidate A, they can force A to win by giving A the highest possible score and every other candidate a zero. However if only a minority of voters vote like this and A loses, then these voters may have missed an opportunity to cause a different candidate they approve of to win (Smith, 2011). Like the Borda count, range voting satisfies the monotonicity criterion because any additional score given to a candidate can only help that candidate win and will have no negative effect on the outcome for the candidate. Range voting is also not affected by cloning since a voter is able to give the clones similar scores. This does not help one of the similar candidates win (like the Borda count) or hurt the chance of the similar candidates (like plurality). Finally, the chance of ties in a range voting election, with either a range of 0-4 or 0-10 is less than the chance of ties for plurality voting.

Section 6: Analysis of Ties

Every voting method investigated in this paper has the possibility of resulting in a tied election. Determining the winner of a tied political election can be very costly. Because of this, it is desirable for a voting method to produce fewer rather than more tied elections. In practice there are both exact ties and apparent ties.

Definition 6.1	**Exact Tie vs. Apparent Tie**
An exact tie is one in which candidates have the same number of votes. An apparent tie is one in which the total votes for two or more candidates are within a certain percentage of each other. For example, the State of Minnesota considers an election tied when the votes for candidates are within ½% of each other. When this happens, the state will pay for a recount.	

In recent years, there have been high profile elections in the United States which have resulted in apparent ties. Costly runoff elections or recounts have been executed to resolve such ties. For instance, the 2008 United States Senate election in Minnesota was very costly.

Example 6.1	**The Cost of Ties in Political Elections**

2008 MN U.S. Senate Election		
Candidate	**Votes Before Recount**	**Votes After Recount**
Al Franken	1,211,375 (41.98%)	1,212,431 (41.991%)
Norm Coleman	1,211,590 (41.988%)	1,212,206 (41.984%)
Dean Barkley	437404 (15.158%)	437,505 (15.153%)

This election resulted in a statistical tie and the percentages for Franken and Coleman were within ½% of each other. Therefore a recount was undertaken, which cost the state of Minnesota both financially (over $12 million) and in terms of time passed without a U.S. Senator (238 days) (StarTribune.com, 2010).

Because recounts are costly, ties have been recognized as a serious problem in elections. This is one of the motivations behind a push for alternatives to plurality voting. In this project, simulated elections were generated using plurality, instant runoff voting and range voting and the number of ties were counted in order to see if these voting methods varied significantly in terms of the probability of tied elections. 10,000 elections were simulated using Mathematica 7.0. Only exact ties were counted, however it is expected that the results for apparent ties should be similar to those of exact ties. Further simulations could be done to investigate this hypothesis. The chance of ties for the Borda count were not analyzed in this project because is the Borda count is not currently used for political elections in the United States, nor does it seem to be a clear improvement over plurality because it suffers from cloning. For range voting, both randomized elections, those where all voters assigned numbers randomly, and normalized elections, those in which voters gave at least one candidate a zero and at least one candidate the highest score possible were simulated. For plurality and range voting, an election is considered a tie when there is a tie for first place. For IRV, an election is considered a tie when no majority winner exists and there is a tie for last place or when the final round of an IRV election results in a tie. More detail on how the simulations were conducted and the simulation code are included in Appendix A2.

Table 6.1 — **Percent of 10,000 Simulated Elections with 3 Candidates and N Voters Resulting in a Tie**

N	**Plurality**	**IRV**	**Randomized Range Voting (0-4)**	**Normalized Range Voting (0-4)**
2	66.6%	66.63% (0%)	19.33%	23.12%
3	22.04%	21.96% (0%)	16.34%	15.13%
5	36.81%	0% (0%)	12.67%	10.05%
10	17.43%	43.26% (7.72%)	9.87%	6.19%
100	7.02%	16.42% (0.63%)	3.03%	2.08%
1,000	2.1%	5.33% (0.08%)	0.93%	0.56%
10,000	0.7%	1.74% (0%)	0.25%	0.27%
100,000	0.18%	0.41% (0%)	0.03%	0.02%

This table lists, for each voting method, the percent of simulated elections resulting in an exact tie for three candidates and N voters. In the IRV column, the first number is the percent of elections that included a tie in at least one round, while the number in parentheses is the percent of elections for which a tie occurred in both rounds.

Table 6.1 shows that for small numbers of voters (N), the frequency of ties for IRV jumps around as N increases. In fact, when N is 5, it is impossible for an IRV election to be tied. When the number of voters increases beyond 100 the frequency of ties for IRV remains higher than for plurality.

When there are 10,000 voters, IRV is over 2.48 more times likely to result in a tie compared to plurality. Range voting is less likely to result in a tie than either of the other two methods (with the exception of IRV for N=5). For 10 voters, IRV is more likely to have elections tied in both rounds, than range voting is to have a tie. For 10,000 voters, plurality is more than 2.59 times, and IRV is more than 6.4 times more likely to result in a tie than is normalized range voting. For 100,000 voters plurality is more than 9 times and IRV is more than 20.5 times more likely than normalized range voting to result in a tie. If only the chance of ties for range voting is examined, some difference between randomized scores and normalized scores can be seen. However, both have less chance of ties than plurality and IRV. The actual chance of a tie in a real range voting election would probably lie somewhere between these two, as some voters would give scores similar to the randomized scores even though all voters would be instructed to normalize their vote in order to give their favorite candidate(s) maximum support over their least favorite candidates.

The results for four candidate elections, summarized in Table 6.2, were similar to the three candidate election results.

Table 6.2 — **Percent of 10,000 Simulated Elections with 4 Candidates and N Voters Resulting in a Tie**

N	Plurality	IRV	Randomized Range Voting (0-4)	Normalized range Voting (0-4)
2	75.32%	75.44% (0%) (0%)	23.34%	19.93%
3	37.32%	37.96% (0%) (0%)	19.03%	16.2%
5	34.71%	23.13% (0%) (0%)	15.19%	11.63%
10	27.94%	76.02% (17.33%) (3.62%)	11.2%	8.55%
100	9.22%	27.24% (2.31%) (0.06%)	3.53%	2.68%
1,000	3.18%	8.65% (0.26%) (0%)	1.03%	0.91%
10,000	0.91%	3.15% (0%) (0%)	0.35%	0.25%
100,000	0.28%	0.83% (0%) (0%)	0.08%	0.08%

This table lists the percent of 10,000 simulated elections with four candidates and N voters that were tied. The number in each column is the percent of ties for each election method. For IRV, the first number is the percent of elections that were tied in at least one round, the number in first set of parentheses is the percent of elections for which ties occurred in exactly two rounds and the number in second set of parentheses is the percent of elections for which ties occurred in all three rounds.

Tables 6.1 and 6.2 summarize simulation results for range voting with the range 0-4. But the choice of range affects the frequency of ties for range voting. Therefore in this project simulations were conducted using a variety of practical ranges. The smallest range used was 0-1 (approval voting) and the largest range used was 0-10. Larger ranges than 0-10 are impractical because they require large ballots (a range voting ballot has a column for each integer value in the range) and because voters tend not to parse their relative approval ratings for candidates more finely than 0-10. Tables 6.2 and 6.3 display how different choices for range sizes affect the percent of ties in a range voting election.

Table 6.3 **Percent of 10,000 Simulated Elections with 3 Candidates and N Voters Resulting in a Tie**

N	Plurality	Randomized Range Voting (0-1)	Normalized Range Voting (0-1)	Randomized Range Voting (0-4)	Normalized Range Voting (0-4)	Randomized Range Voting (0-10)	Normalized Range Voting (0-10)
2	66.6%	48.84%	48.84%	19.33%	23.12%	8.99%	15.33%
3	22.04%	40.15%	40.15%	16.34%	15.13%	7.57%	6.22%
5	36.81%	33.43%	33.43%	12.67%	10.05%	5.65%	4.12%
10	17.43%	24.55%	24.55%	9.87%	6.19%	3.99%	2.32%
100	7.02%	8.3%	8.3%	0.03%	2.08%	1.08%	0.61%
1,000	2.1%	2.4%	2.4%	0.93%	0.56%	0.58%	0.16%
10,000	0.7%	0.73%	0.73%	0.25%	0.27%	0.18%	0.05%
100,000	0.18%	0.25%	0.25%	0.03%	0.02%	0.05%	0%

This table lists the percent of 10,000 simulated elections with 3 candidates and N voters that were tied. It shows results for both randomized and normalized range voting for a variety of ranges.

Table 6.4	Percent of 10,000 Simulated Elections with 4 Candidates and N Voters Resulting in a Tie

N	Plurality	Randomized Range Voting (0-1)	Normalized Range Voting (0-1)	Randomized Range Voting (0-4)	Normalized Range Voting (0-4)	Randomized Range Voting (0-10)	Normalized Range Voting (0-10)
2	75.32%	55.5%	55.5%	23.34%	19.93%	11.45%	13.88%
3	37.32%	48.27%	48.27%	19.03%	16.2%	8.91%	11.08%
5	34.71%	37.99%	37.99%	15.19%	11.63%	6.77%	8.04%
10	27.94%	28.55%	28.55%	11.2%	8.55%	4.74%	6.05%
100	9.22%	10%	10%	3.53%	2.68%	1.6%	1.94%
1,000	3.18%	3.17%	3.17%	1.03%	0.91%	0.55%	0.61%
10,000	0.91%	1.17%	1.17%	0.35%	0.25%	0.19%	0.22%
100,000	0.28%	0.26%	0.26%	0.08%	0.08%	0.06%	0.05%

This table lists the percent of 10,000 simulated elections with 4 candidates and N voters that were tied. It shows results for both randomized and normalized range voting for a variety of ranges.

As summarized in Tables 6.3 and 6.4, as the size of the range increases, the frequency of ties for simulated range-voting elections decreases Note that for range 0-1 (approval voting) , the chance of a tie is somewhat greater than that of plurality. Therefore when selecting a range to use, it would be beneficial to both have a range large enough to reduce the chance of ties below that of plurality, but not so large that ballots are cumbersome and voters are confused by the range. Of the ranges discussed in this paper, 0-4 seems to be the most practical for use in political elections.

From these simulations, one can predict that IRV elections with three or four candidates will result in more ties than plurality elections. In contrast, the frequency of ties for range voting would be less than for plurality if the range is large enough. In these simulations a range of 0-4 is more than large enough.

Section 7: Conclusions

In this report, examples illustrated how plurality voting, IRV and the Borda count fall short of ideal by violating the IIA and CWC criteria of Arrow's Impossibility Theorem. Interpretations of IIA and CWC were further explored for range voting, which is a method that does not use rank-order ballots. This report found that the voting method goodness criteria known as monotonicity and clone independence are highly desirable for political elections. It is also beneficial for a voting method to utilize more information from each voter than plurality does in order to better capture the will of the people. Another aspect of voting methods to take into consideration is the frequency of tied elections since these are costly for political elections.

Each of the voting methods discussed displays some kinds of draw backs. Plurality utilizes the least information from voters as it only takes into account each voter's first place preference. It also does not satisfy IIA, CWC, or clone independence. However, it is widely used for political elections in the United States because it is simple for voters to understand, it is easy for officials to tabulate, and it has the most extensive history of use in the United States.

IRV is clone independent and allows each voter to submit more information regarding preference for candidates than does plurality. This is in the form of a ranked ordered ballot. However, IRV does not utilize every voter's ballot to the same extent as only the first preference of some voters is counted while all the preference information of other votes' ballots may be used. IRV also has a significant problem, non-monotonicity, that plurality does not have. Non-monotonicity is a major flaw because it leads to unpredictable election result. It can have unintended consequences for voters since giving a favorite candidate a first place preference ranking can either help our hurt this candidates chances of winning, depending on the circumstances. IRV elections also have a higher probability of ties than plurality elections.

The Borda count is superior to both plurality and IRV in the sense that it utilizes the full preference list ballot of every voter. Also, like plurality, the Borda count is monotonic. Unfortunately, the Borda count has its own unique problem, that of teaming. Candidates are able to influence the election in their favor by encouraging other similar candidates to run. Teaming would likely be exploited in political elections.

Range voting is monotonic like plurality and the Borda count and it is immune to candidate cloning, as is IRV. It utilizes every voter's ballot to the same extent and range voting ballots can allows voters to be even more expressive than IRV and the Borda count when the range is large enough, as each voter gives an independent score to each of the candidates. Arrow's Impossibility Theorem does not apply to range voting since it is not a ranked ordered voting method. Numerical simulations indicated that the chance of a tied election decreases as the size of the range used in range voting increases. Using a 0-4 range greatly reduces the chance of ties for a range voting election compared to a plurality voting election.

If a city or county were considering changing voting methods, this report would recommend range voting with a 0-4 range. This range allows the voters to be expressive, while not creating cumbersome ballots. The one major drawback of range voting found in this project is that it has never been used for political elections. So range voting should be conducted for test elections, and be deemed constitutional before it is used for political elections.

Section 8: Recommendation for Further Study

The following issues and questions were encountered during the course of creating this report. They were beyond the scope of this project but could be investigated in further research.

Multiple Winner Elections

IRV supporters are proposing that IRV be used for elections with multiple winners. An in depth analysis of the use of IRV for multiple-winner elections was beyond the scope of this project. However, it was determined that generalizing IRV from single-winner elections to multiple-winner elections is not straight forward. For instance, when using IRV to elect multiple candidates, say there will be W winners, any candidate who receives more than 1/(W+1) of the votes wins. When a candidate receives more than 1/(W+1) of the votes, the candidate's excess votes are then transferred proportionally to the other candidates. For example, in an election with 3 winners, suppose that in the first round candidate A gets 35% of the vote. Since 1/(3+1) is 25%, then 35%-25% or 10% of the vote for A would be transferred to other candidates. However, whether this transfer of votes is done before or after the first round elimination can affect the election results. A last place candidate, say B, could benefit from this transfer and as a result a different candidate, say C, could be eliminated. However if the elimination happens before the vote transfer, candidate B is eliminated. Simulations of the elimination process for multiple-winner IRV elections could be carried out to investigate how frequently the timing of the excess-vote transfer affects election outcomes.

Two-Party Domination

Two party domination narrows the choices for voters and in this sense is undemocratic. Duverger's Law is a principle which states that plurality voting tends to favor a two-party system (Wikipedia, 2010). At first glance many people think that IRV will not lead to two-party domination. However, Australia has a long track-record of using IRV and having elections dominated by two parties. Since 1923, Australia has only elected a president from one of the 2 major parties (Kok, 2011). Could the non-monotonic, chaotic behavior of IRV entrain voters over time to always mark as top choice a candidate from one of two dominant parties? A future study could investigate the hypothesis that IRV tends to reinforce two-party domination.

Hybrid Election Methods

No current election method is perfect; however research could be done in order to see the effects of combining two or more election method to improve on each individual method. Condorcet's method is widely believed to be the best method when it produces a winner. This could be a good election method to implement in combination with another voting method that would be used to determine a winner when no Condorcet winner exists. One could investigate which method or methods would perform best in combination with Condorcet's method.

Further Study of Range Voting

Range voting has recently entered the national discussion of voting methods for political elections. There are many research topics pertaining to range voting that could be investigated. Relating to this report, one could test and/or prove Conjecture 5.1 (Intensity of Independent of Irrelevant Alternatives (IIIA)) and Conjecture 5.2 (Intensity of Condorcet Winner Criterion (ICWC)). Also because range voting has not been used in political elections, one could conduct small-scale elections using range voting to see how the method works in practice and how voters react to the method.

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A2. Code for Mathematica Simulations

A2.1 Three Candidate Elections

```
(*Plurality Voting created with Wolfram Mathematica 7.0

3 Candidates Election Simulated m times for n voters

For each voter, a random number generator picks one of the candidates
(x,y,z).  One vote is added to the candidate (a,b,c) that is randomly
chosen.  This continues for the desired number of voters (n).  Once
the election is over, it checks for a first place tie between any pair
of candidates and a three way tie between the candidates.  A counter
(ab,ac,bc,abc) then gets incremented for each type of tie if one
exists.  This is done for the desired amount of elections (m).  In the
end the number of ties is divided by the number of elections in order
to calculate the probability of ties.
*)


m=10000(*number of elections to simulate*)
n=1000(*number of voters*)
ab=0;ac=0;bc=0;abc=0

For[j=0,j<m,j++,

 a=0;b=0;c=0;
 For[i=0,i<n,i++;
  s=RandomInteger[{0,2}];
  If[s□0,x=1;y=0;z=0];
  If[s□1, y=1;x=0;z=0];
  If[s□2,z=1;x=0;y=0];

  a=a+x;b=b+y;c=c+z;
  ];
 If[a□b&&b>c,ab++];
 If[a□c&&c>b, ac++];
 If[b□c&&c>a,bc++];
 If[a□b□c,abc++]
 ]
Prob=(ab+ac+bc+abc)/m
N[%]
```

```
(*Instant Runoff Voting created with Wolfram Mathematica 7.0

3 Candidate Elections Simulated m times for n voters

For each voter, a random number generator picks one of the candidates
(x,y,z).  One vote is added to the candidate (a,b,c) that is randomly
chosen.  This continues for the desired number of voters (n).  The
candidates are then sorted from most votes reviewed to least and each
one is assigned a new variable (a1,b1,c1).  Now if candidate a1 (the
highest) has over half of the votes, the election is over and no tie
exists.  Otherwise, it checks for a tie in last place, if there is,
then R1 is incremented.  Now candidate c1 (the lowest) is eliminated
and a loop gets run through for each vote c1 received, redistributing
votes to a1 and b1.  Now it checks for a tie between the remaining
candidates and increments R2 if there is a tie.  For each round there
is another variable r1 (round 1) and r2 (round 2), if there is a tie
in round 1, r1=1 and likewise for round 2. At the end of the election,
if both of these are 1, there was a tie in both rounds and R12 is
incremented.  To find the number of elections that had ties the number
ties for each round must be added and then the number of times two
ties occurred in the same election is subtracted. (I am concerned with
how many elections have ties, not how many ties happen in 10000
elections.  Double counting ties may lead to deceiving results.) In
the end the number of ties is divided by the number of elections in
order to calculate the probability of ties. *)

m=10000(*number of elections to simulate*)
n=1000(*number of voters*)
R1=0;R2=0;R12=0;

For[j=0,j<m,j++,
 a=0;b=0;c=0;r1=0;r2=0;
 For[i=0,i<n,i++;s=RandomInteger[{0,2}];
  If[s□0,x=1;y=0;z=0];
  If[s□1,x=0;y=1;z=0];
  If[s□2,x=0;y=0;z=1];

  a=a+x;b=b+y;c=c+z];

 L1=Sort[{a,b,c},Greater];
 a1=L1[[1]];b1=L1[[2]];c1=L1[[3]];

 If[a1<=n/2,
  If[b1□c1,R1=R1+1;r1=1];
  For[p=c1,p>0,p=p-1;g=RandomInteger[{0,1}];
   If[g□0,x=1;y=0];
   If[g□1,x=0;y=1];
   a1=a1+x;b1=b1+y];
  L2=Sort[{a1,b1},Greater];
  a2=L2[[1]];b2=L2[[2]];
```

```
    If[a2□b2,R2=R2+1;r2=1]
    ];
  If[r1==r2□1, R12=R12+1]
  ]
Ptie=N[(R1+R2-R12)/m]
Print[Ptie]
Print[N[R12/m]]
```

```
(*Random Range Voting created with Wolfram Mathematica 7.0

3 Candidate Elections Simulated m times for n voters

For each voter, a random number generator picks a score from 0 to max
(x,y,z) for each candidate.  This score is added to the candidates
total (a,b,c).  This continues for the desired number of voters (n).
Once the election is over it checks for a first place tie between any
pair of candidates and a three way tie between the candidates.
Counters (ab,ac,bc,abc)  get incremented for each type of tie if one
exists.  This is done for the desired amount of elections (m).  In the
end the number of ties is divided by the number of elections in order
to calculate the probability of ties.
*)


m=10000(*number of elections to simulate*)
n=10(*number of voters*)
max=10(*maximum score a voter can give*)
ab=0;ac=0;bc=0;abc=0

For[j=0,j<m,j++;
 a=0;b=0;c=0;
 For[i=0,i<n,i++;
  x=RandomInteger[max];
  y=RandomInteger[max];
  z=RandomInteger[max];
  a=a+x;b=b+y;c=c+z
  ];
 If[a□b&&b>c,ab++];
 If[a□c&&c>b, ac++];
 If[b□c&&c>a,bc++];
 If[a□b□c,abc++]
 ]

Prob=(ab+ac+bc+abc)/m
N[%]
```

```
(*Normalized Range Voting created with Wolfram Mathematica 7.0

3 Candidates Election Simulated m times for n voters

For each voter, a random number generator picks a score from 0 to max
(x,y,z) for each candidate. These scores are compared. If they are
equal, they are left the same. Otherwise, the lowest score (or scores
if there is a tie) is changed to 0, and highest score is changed to 4
(or scores if there is a tie). If there is no tie, the middle score
stays the same. These new numbers for x,y,z, are added to the running
totals for each of the candidates (a,b,c).  This continues for the
desired number of voters (n). Once the election is over, it checks for
a first place tie between any pair of candidates and a three way tie
between the candidates. A counter (ab,ac,bc,abc) then gets incremented
for each type of tie if one exists. This is done for the desired
amount of elections (m).  In the end the number of ties is divided by
the number of elections in order to calculate the probability of ties.
*)
 m=10000(*number of elections to simulate*)
n=1000(*number of voters*)
max=4(*maximum score a voter can give*)
ab=0;ac=0;bc=0;abc=0
For[j=0,j<m,j++,

 a=0;b=0;c=0;

For[i=0,i<n,i++;x=RandomInteger[max];y=RandomInteger[max];z=RandomInte
ger[max];
  If[x□y>z,x=max;z=0;y=max];
  If[x□z>y, x=max;y=0;z=max];
  If[y==z>x,y=max;x=0;z=max];
  If[x□y<z,z=max;x=0;y=0];
  If[x□z<y, y=max;x=0;z=0];
  If[y□z<x,x=max;y=0;z=0];
  If[x>y>z,x=max;z=0];
  If[x>z>y,x=max;y=0];
  If[y>x>z,y=max;z=0];
  If[y>z>x,y=max;x=0];
  If[z>y>x,z=max;x=0];
  If[z>x>y,z=max;y=0];
  a=a+x;b=b+y;c=c+z;
  ];
 If[a□b&&b>c,ab++];
 If[a□c&&c>b, ac++];
 If[b□c&&c>a,bc++];
 If[a□b□c,abc++]
 ]

Prob=(ab+ac+bc+abc)/m
N[%]
```

A2.2 Four Candidate Elections

The coding for the four candidate elections were done using algorithms similar to those used for the three candidate elections. The algorithms used for the three candidate elections were modified slightly to accommodate a fourth candidate. For further descriptions of the algorithms, see Appendix A2.1.

```
(*Plurality Voting created with Wolfram Mathematica 7.0

4 Candidates Election Simulated m times for n voters*)



m=10000(*number of elections to simulate*)
n=3(*number of voters*)

tie=0

For[j=0,j<m,j++,
 a=0;b=0;c=0;d=0;

 For[i=0,i<n,i++;s=RandomInteger[{0,3}];
  If[s□0,w=1;x=0;y=0;z=0];
  If[s□1,w=0;x=1;y=0;z=0];
  If[s□2,w=0;x=0;y=1;z=0];
  If[s□3,w=0;x=0;y=0;z=1];
  a=a+w;b=b+x;c=c+y;d=d+z
  ];

 L=Sort[{a,b,c,d},Greater];
 If[L[[1]]□L[[2]],tie++]
 ]

Print[ tie]
Prob=(tie)/m
N[%]
```

```
(*Instant Runoff Voting created with Wolfram Mathematica 7.0

4 Candidate Elections Simulated m times for n voters*)

m=10000(*number of elections to simulate*)
n=1000(*number of voters*)

R1=0;R2=0;R3=0;R12=0;R13=0;R23=0;R123=0;
For[j=0,j<m,j++,
 a=0;b=0;c=0;d=0;r1=0;r2=0;r3=0;
 For[i=0,i<n,i++;s=RandomInteger[{0,3}];
  If[s□0,w=1;x=0;y=0;z=0];
  If[s□1,w=0;x=1;y=0;z=0];
  If[s□2,w=0;x=0;y=1;z=0];
  If[s□3,w=0;x=0;y=0;z=1];

  a=a+w;b=b+x;c=c+y;d=d+z];

 L1=Sort[{a,b,c,d},Greater];

 a1=L1[[1]];b1=L1[[2]];c1=L1[[3]];d1=L1[[4]];

 If[a1<=n/2,

   If[c1□d1,R1=R1+1;r1=1];

   For[l=d1,l>0,l=l-1;g=RandomInteger[{0,2}];
    If[g□0,w=1;x=0;y=0];
    If[g□1,w=0;x=1;y=0];
    If[g□2,w=0;x=0;y=1];
    a1=a1+w;b1=b1+x;c1=c1+y];

   L2=Sort[{a1,b1,c1},Greater];

   a2=L2[[1]];b2=L2[[2]];c2=L2[[3]];
   If[a2<=n/2,

    If[b2□c2,R2=R2+1;r2=1];

    For[p=c2,p>0,p=p-1;g=RandomInteger[{0,1}];
     If[g□0,w=1;x=0];
     If[g□1,w=0;x=1];
     a2=a2+w;b2=b2+x];

    L3=Sort[{a2,b2},Greater];

    a3=L3[[1]];b3=L3[[2]];

    If[a3□b3,R3=R3+1;r3=1];
    ]]
```

```
  If[r1□r2□r3□1, R123=R123+1,If[r1□r2□1, R12=R12+1];If[r1□r3□1,
R13=R13+1];If[r2□r3□1, R23=R23+1]];
 ]
Ptie=N[(R1+R2+R3-R12-R13-R23+R123)/m];
Print[Ptie]
Print[N[(R12+R13+R23-R123)/m]]
Print[N[R123/m]]
```

```
(*Random Range Voting created with Wolfram Mathematica 7.0

4 Candidate Elections Simulated m times for n voters*)

m=10000(*number of elections to simulate*)
n=2(*number of voters*)
max=4(*maximum score a voter can give*)
tie=0
For[j=0,j<m,j++,

 a=0;b=0;c=0;d=0;

For[i=0,i<n,i++;w=RandomInteger[max];x=RandomInteger[max];y=RandomInte
ger[max];z=RandomInteger[max];(*Print[x,y,z];*)

  a=a+w;b=b+x;c=c+y;d=d+z

  ];
 L=Sort[{a,b,c,d},Greater];
 If[L[[1]]□L[[2]],tie++]

 ]
Print[ tie]
Prob=(tie)/m
N[%]
```

```
(*Normalized Range Voting created with Wolfram Mathematica 7.0

4 Candidates Election Simulated m times for n voters*)

m=10000;(*number of elections to simulate*)
n=2(*number of voters*)

max=4;(*maximum score a voter can give*)
all=0;ab=ac=ad=bc=bd=cd=abc=abd=acd=bcd=abcd=0;
For[j=0,j<m,j++,

 a=0;b=0;c=0;d=0;

For[i=0,i<n,i++;w=RandomInteger[max];x=RandomInteger[max];y=RandomInte
ger[max];z=RandomInteger[max];(*Print[w,x,y,z];*)

   L=Sort[{w,x,y,z},Less];

   If[L[[1]]□L[[2]]□L[[3]]□L[[4]],all++ ,
    If[L[[1]]□L[[2]]□L[[3]], If[L[[4]]==w, w=4;x=y=z=0,If[L[[4]]□x,
x=4;w=y=z=0, If[L[[4]]□y, y=4;w=x=z=0,z=4;w=x=y=0]]],
     If[L[[2]]□L[[3]]□L[[4]], If[L[[1]]==w, w=0;x=y=z=4,If[L[[1]]□x,
x=0;w=y=z=4, If[L[[1]]□y, y=0;w=x=z=4,z=0;w=x=y=4]]],
      If[L[[1]]□L[[2]]&&L[[3]]≠ L[[4]],If[w==L[[1]]||L[[2]],w=0];
       If[x==L[[1]]||L[[2]],x=0];
       If[y==L[[1]]||L[[2]],y=0];
       If[z==L[[1]]||L[[2]],z=0];
       If[w==L[[4]],w=4];
       If[x==L[[4]],x=4];
       If[y==L[[4]],y=4];
       If[z==L[[4]],z=4],
       If[L[[1]]≠L[[2]]&&L[[3]]== L[[4]],
        If[w==L[[1]],w=0];
        If[x==L[[1]],x=0];
        If[y==L[[1]],y=0];
        If[z==L[[1]],z=0];
        If[w==L[[3]]||L[[4]],w=4];If[x==L[[3]]||L[[4]],x=4];
        If[y==L[[3]]||L[[4]],y=4];If[z==L[[3]]||L[[4]],z=4],
        If[L[[1]]□L[[2]]&&L[[3]]== L[[4]],If[w==L[[1]]||L[[2]],w=0];

If[x==L[[1]]||L[[2]],x=0];If[y==L[[1]]||L[[2]],y=0];If[z==L[[1]]||L[[2
]],z=0];If[w==L[[3]]||L[[4]],w=4];If[x==L[[3]]||L[[4]],x=4];If[y==L[[3
]]||L[[4]],y=4];
         If[z==L[[3]]||L[[4]],z=4],
         If[w==L[[1]],w=0];
         If[x==L[[1]],x=0];
         If[y==L[[1]],y=0];
         If[z==L[[1]],z=0];
         If[w==L[[4]],w=4];
         If[x==L[[4]],x=4];
```

```
        If[y==L[[4]],y=4];
        If[z==L[[4]],z=4]
        ]]]]]];


  a=a+w;b=b+x;c=c+y;d=d+z


  ];
 If[a□b&&b>c&&b>d,ab++];
 If[a□c&&c>b&&c>d, ac++];
 If[a□d&&d>b&&d>c, ad++];
 If[b□c&&c>a&&c>d,bc++];
 If[b□d&&d>a&&d>c,bd++];
 If[c□d&&d>a&&d>b,cd++];
 If[a□b□c&&c>d,abc++];
 If[a□b□d&&d>c,abd++];
 If[b□c□d&&c>a,bcd++];
 If[a□c□d&&c>b,acd++];
 If[a□b□c==d,abcd++]
 ];

Prob=N[(ab+ac+ad+bc+bd+cd+abc+abd+acd+bcd+abcd)/m];
Print["Strategic Range Vote with ",n," voters:",Prob ];
```