Optimal Spring-Damper Location for a Beam-Spring System

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Abstract

This paper studies the solution behavior of a system of two coupled elastic beams connected vertically by springs. With the viscous damping collocated with the ends of the springs, the system energy decays exponentially. Our goal is to find the optimal spring-damper location which yields the best energy decay rate of the beam-spring system.
1 Introduction

Our main interest comes from a system with two elastic beams connected by vertical springs, which could be represented by the following partial differential equations:

\[
\begin{align*}
\frac{\partial^2 u}{\partial t^2} &= -a_1 \frac{\partial^4 u}{\partial x^4} - k(x)(u-y) - \delta k(x)u_t \\
\frac{\partial^2 y}{\partial t^2} &= -a_2 \frac{\partial^4 y}{\partial x^4} + k(x)(u-y) - \delta k(x)y_t \\
u(0, t) &= u(\pi, t) = u_{xx}(0, t) = u_{xx}(\pi, t) = 0 \\
u(x, 0) &= u(x, 0) = u_1(x), y(x, 0) = y_0(x), y_t(x, 0) = y_1(x)
\end{align*}
\]

(1.1) (1.2) (1.3) (1.4)

where \( u(x, t), y(x, t) \) are the displacements of upper and lower beams; \( k(x) \) is the spring location,

\[
k(x) = \begin{cases} 
0 & x \notin (p-s, p+s) \\
1 & x \in (p-s, p+s)
\end{cases}
\]

\( p \in (0, \pi) \) is the center of the damper location, \( 2s \) is the width of the damper; \( \delta \) and \( k(x) \) is the damping coefficient.

In the system (1.1)-(1.4), equation (1.1) describes the vibration of the upper elastic beam and equation (1.2) describes the vibration of the lower elastic beam. Equation (1.3) is the boundary conditions of the two beams. (1.4) is the initial conditions. Here, we assume that the beams are simply supported at \( x = 0 \) and \( x = \pi \). Our goal is to find the optimal locations of the springs and dampers which yields the best energy decay rate of the beam spring system.

1.1 Definitions and Theorems

In this section, we will give a list of definitions and theorem which are going to be used in this paper.

**Definition 1.1** A Norm is a function defined on a vector space \( V \), normally using \( || \cdot || \) to represent it. The norm satisfies the following properties for \( \forall x, y \in V \).

1. \( ||x|| \rightarrow \mathbb{R} \)
2. \( ||ax|| = |a| ||x|| \)
3. \( ||x + y|| \leq ||x|| + ||y|| \)
4. If \( ||x|| = 0 \) then \( x \) is the zero vector.

**Definition 1.2** A Hilbert space \( H \) is a real or complex inner product space that is also a complete metric space with respect to the distance function induced by the inner product. For a complex inner product space, \( H \) is equipped an inner product \( \langle x, y \rangle \) associating a complex number to each pair of elements \( x, y \) of \( H \). The inner product satisfies the following properties.

1. \( \langle x, y \rangle = \langle y, x \rangle \)
2. \( \langle a_1 x_1 + a_2 x_2, y \rangle = a_1 \langle x_1, y \rangle + a_2 \langle x_2, y \rangle \)
3. \( \langle x, x \rangle \geq 0 \)
4. \( \langle x, x \rangle = ||x||^2 \)

**Definition 1.3** The eigenvalues of matrix \( A \) in Hilbert space \( H \) are convergent if

\[
\lim_{N \rightarrow \infty} \lambda_N = \lambda
\]

where \( \lambda \) is an eigenvalue of matrix \( A \) in Hilbert space \( \lambda_N \) is the eigenvalue of matrix \( A_N \) which is the \( N \) dimensional approximation of \( A \).
Definition 1.4 A family $S(t)(0 \leq t < \infty)$ of bounded linear operators in a Banach space $H$ is called a strongly continuous semigroup (in short, a $C_0$ semigroup) if
1. $S(t_1 + t_2) = S(t_1)S(t_2), \forall t_1, t_2 > 0$
2. $S(0) = I$
3. For each $x \in H$, $S(t)x$ is continuous in $t$ on $[0, \infty)$.

For such a semigroup $S(t)$, we define an operator $A$ with domain $D(A)$ consisting of points $x$ such that the limit
$$Ax = \lim_{h \to 0} \frac{S(h)x - x}{h}, \quad x \in D(A)$$
exists. Then $A$ is called the infinitesimal generator of the semigroup $S(t)$. Given an operator $A$, if $A$ coincides with the infinitesimal generator of $S(t)$, then we say that it generates a strongly continuous semigroup $S(t), \ t \geq 0$. Sometimes we also denote $S(t)$ by $e^{At}$.

Definition 1.5 A $C_0$ semigroup $e^{At}$ is stable on $H$ if
$$\lim_{t \to \infty} \|e^{At}x\| = 0, \text{ for } \forall x \in H$$

Definition 1.6 A $C_0$ semigroup $e^{At}$ is said to be exponentially stable if there exist positive constants $M \geq 1, \omega < 0$ such that $\|e^{At}\| \leq Me^{\omega t}, $ for $\forall t \geq 0$.

Here, $\omega$ is constant.

Definition 1.7 A linear evolution equation
$$\{ \dot{Z} = AZ, \quad Z(0) = Z_0 \}$$
is well posed on a Hilbert space, if $A$ generates an associated $C_0$ semigroup $e^{At}$. The unique solution of the evolution equation is
$$Z(t) = e^{At}Z_0$$

Definition 1.8 The solution of evolution equation is said to be exponential stable if there exist $M \geq 1, w < 0$ such that $\|e^{At}\| \leq Me^{\omega t}, $ for $\forall t \geq 0$ and $\forall Z_0 \in H$.

Definition 1.9 Suppose a semigroup $e^{At}$ is exponentially stable. Its growth rate is defined as $\omega_0 = \min\{\omega | \|e^{At}\| \leq Me^{\omega t}, \omega < 0\}$

Definition 1.10 Let $\sigma(A)$ be the spectrum of an operator $A$ on a Hilbert space, and denote
$$r_0 = \max\{\text{Re} \lambda | \lambda \in \sigma(A)\}$$
We say system satisfies the spectrum determined growth property if $r_0 = w_0$.

1.2 Abstract Evolution Equation

Let’s first convert system (1.1)-(1.4) into a first-order abstract evolution equation on a Hilbert space.

Multiplying equation (1.1) by $u_t$, then integrating both sides on $[0, \pi]$, we have
$$\int_0^\pi u_{tt}u_tdx = -a \int_0^\pi u_{xxx}u_tdx - \int_0^\pi k(x)(u - y)u_tdx - \delta \int_0^\pi k(x)u_t^2dx \quad (1.5)$$

This can be further written as
$$\frac{1}{2} \frac{d}{dt} \int_0^\pi u_t^2dx = -\frac{1}{2} \frac{d}{dt} \int_0^\pi au_x^2dx - \int_0^\pi k(x)(u - y)u_tdx - \delta \int_0^\pi k(x)u_t^2dx \quad (1.6)$$
Similarly, we apply the procedure to equation (1.2)

\[
\int_0^\pi y_t y_t dx = -a \int_0^\pi y_{xxxx} y_t dx + \int_0^\pi k(x)(u - y)y_t dx - \delta \int_0^\pi k(x)y_t^2 dx \tag{1.7}
\]

which further leads to

\[
\frac{1}{2} \frac{d}{dt} \int_0^\pi y_t^2 dx = -\frac{1}{2} \frac{d}{dt} \int_0^\pi ay_t^2 dx + \int_0^\pi k(x)(u - y)y_t dx - \delta \int_0^\pi k(x)y_t^2 dx. \tag{1.8}
\]

Hence, the sum of (1.6) and (1.8) gives

\[
\frac{1}{2} \frac{d}{dt} \int_0^\pi (u_t^2 + y_t^2 + a u_{xx}^2 + ay_{xx}^2 + k(x)(u - y)^2) dx = -\delta \int_0^\pi (k(x)u_t^2 + k(x)y_t^2) dx. \tag{1.9}
\]

Denote

\[
E(t) = \frac{1}{2} \int_0^\pi (u_t^2 + y_t^2 + a u_{xx}^2 + ay_{xx}^2 + k(x)(u - y)^2) dx \tag{1.10}
\]

where \(\frac{1}{2}(u_t^2 + y_t^2)\) represents the kinetic energy of the two beams; \(\frac{1}{2}a(u_{xx}^2 + ay_{xx}^2)\) represents the potential energy of the two beams; \(\frac{1}{2}k(x)(u - y)^2\) represents the potential energy of the springs. Then, equation (1.9) represents the changing rate of the energy of the elastic system. From the right hand side of equation (1.9), we see that the changing rate of energy is negative if \(\delta > 0\), the energy of the system would keep decreasing as the time \(t\) increases; the system is conservative if \(\delta = 0\), i.e., \(E(t)\) is a constant.

Define a Hilbert Space \(H = U \times V \times Y \times W\), where

\[
U = (H^1_0(0, \pi) \cap H^2(0, \pi)), \quad V = L^2(0, \pi), \quad Y = (H^1_0(0, \pi) \cap H^2(0, \pi)), \quad W = L^2(0, \pi)
\]

and

\[
L^2(0, \pi) = \{ f(x) | \int_0^\pi f^2(x) dx < \infty \},
\]

\[
H^k(0, \pi) = \{ f(x)|f(x), f'(x)\ldots f^{(k)}(x) \in L^2(0, \pi) \},
\]

\[
H^1_0(0, \pi) = \{ f(x) \in H^1(0, \pi) | f'(0) = f'(\pi) = 0 \}.
\]

For any which \(Z = (u, v, w, w)^T \in H\), \(Z_1 = (u_1, v_1, y_1, w_1)^T \in H\), the Hilbert Space \(H\) has inner product

\[
(Z, Z_1) = ((u, v, w, w)^T,(u_1, v_1, y_1, w_1)^T) = a<u_{xx}, u_{1,xx}>_{L^2} + a<y_{xx}, y_{1,xx}>_{L^2}
\]

\[
+<v, v_1>_{L^2} + <w, w_1>_{L^2} + <k(x)(u - y), u_1 - y_1>_{L^2} \tag{1.11}
\]
with the usual $L^2$ inner product,

$$<f, g>_{L^2} = \int_0^\pi f(x)g(x)dx.$$  

Note that $\|Z\|^2 = 2E(t)$.

Introducing $v = u_t$ and $w = y_t$, equations (1.1) and (1.2) then can be formulated as a first order system

$$\begin{cases}
\frac{du}{dt} = v \\
\frac{dv}{dt} = -au_{xxxx} - k(x)(u - y) - k(x)\delta v \\
\frac{dw}{dt} = w \\
\frac{dw}{dt} = -ay_{xxxx} + k(x)(u - y) - k(x)\delta w
\end{cases}.$$  

From here, we use dot (i.e $\dot{x}$) to represent the time derivative.

Let $Z = (u, v, y, w)^T$, $Z_0 = (u_0, v_0, y_0, w_0)^T$. Then system (1.12) could be rewritten as

$$\begin{cases}
\dot{Z} = AZ \\
Z(0) = Z_0
\end{cases}.$$  

Here, the operator $A : D(A) \subset H \to H$ is defined as

$$A = \begin{bmatrix}
0 & I & 0 & 0 \\
-aD^4 - k(x) & -k(x)\delta & k(x) & 0 \\
0 & 0 & 0 & I \\
k(x) & 0 & -aD^4 - k(x) & -k(x)\delta
\end{bmatrix}.$$  

$$D(A) = \{Z \in H|u, y \in H^4(0, \pi); v, w \in H^1_0(0, \pi) \cap H^2(0, \pi); u''(0) = u''(\pi) = y''(0) = y''(\pi) = 0\}.$$  

The following two theorems can be found in the reference [2]:

**Theorem 1.11** The operator $A$ generates a $C_0$-semigroup of construction, $e^{At}$, on the Hilbert space $H$. Therefore, system is well-posed, i.e, for any $Z_0 \in H$, the unique solution to the system is $Z(t) = e^{At}Z_0, t > 0$.

**Theorem 1.12** The semigroup $e^{At}$ is exponentially stable.

Moreover, the next theorem can be proved based on reference [3].

**Theorem 1.13** The system (1.13) satisfies the spectrum determined growth property.
1.3 Goals of the Project

Our goal is to analyze how the locations of the collocated springs and the dampers influence the decay rate of the energy of the elastic system. However, the growth rate $\omega_0$ is very difficult to compute based on its definition, not mention that in our case it also depends on the damper location $p$. Fortunately, our system satisfies the spectrum determined growth property. We can compute $r_0(p)$ instead of $\omega_0(p)$ for each location $p$. The corresponding eigenvalue problem is

$$
\begin{align*}
\lambda^2 u &= -au''' + k(x)(u - y) - \delta k(x)u \\
\lambda^2 y &= -ay''' - k(x)(u - y) - \delta k(x)y \\
u(0) &= u(\pi) = u''(0) = u''(\pi) = 0, \\
y(0) &= y(\pi) = y''(0) = y''(\pi) = 0.
\end{align*}
$$

(1.16)

If an explicit expression of $r_0(p)$ could be derived from the above eigenvalue problem, then we would be able to find the minimum of $r_0(p)$ over all admissible $p$. However, this is another difficult task.

To get around these difficulties, we will take a practical approach to compute $r_0(p)$ and $\min r_0(p)$. First, we choose a sequence of finite dimensional subspace $H_N$ of $H$ such that $\lim_{N \to \infty} H_N = H$. Next, we project system (1.12) onto $H_N$ to get

$$
\begin{align*}
\dot{Z}_N &= A_N Z_N \\
Z_N(0) &= Z_N(0).
\end{align*}
$$

(1.17)

The eigenvalues of the matrix $A_N$ can be computed numerically, which leads to $r_0^N(p)$, the maximum of the real part of all eigenvalues for a given $p$.

Finally, we compute $r_0^N(p)$ at a set of mesh points of $p$ to obtain the optimal locations of the dampers.

2 Finite Dimensional Approximation

In order to construct a finite dimensional approximation of the infinite dimensional system, we first define a finite dimensional subspace $H_N$ of $H$. Then, project the system onto this space.

2.1 Finite Dimensional Space $H_N$

We define a finite dimensional subspace $H_N$ of $H$ by

$$
H_N = U^N \times V^N \times Y^N \times W^N
$$

where

$$
\begin{align*}
U^N &= \text{span} \left\{ \sqrt{\frac{2}{\pi i^2}} \sin(ix) \right\}_{i=1}^N, \\
V^N &= \text{span} \left\{ \sqrt{\frac{2}{\pi}} \sin(ix) \right\}_{i=1}^N, \\
Y^N &= \text{span} \left\{ \sqrt{\frac{2}{\pi i^2}} \sin(ix) \right\}_{i=1}^N,
\end{align*}
$$

and

$$
\begin{align*}
W^N &= \text{span} \left\{ \sqrt{\frac{2}{\pi}} \sin(ix) \right\}_{i=1}^N.
\end{align*}
$$
\[ W^N = \text{span} \left\{ \frac{\sqrt{2}}{\pi} \sin(ix) \right\}_{i=1}^{N}. \]

The truncated Fourier sine series is chosen because it satisfies the simply supported boundary conditions. Therefore, a basis of \( H_N \) could be written as below:

\[
\begin{align*}
\left\{ \begin{pmatrix}
\sqrt{\frac{2}{\pi}} \frac{1}{i} \sin(x) \\
0 \\
0
\end{pmatrix}, \begin{pmatrix}
\sqrt{\frac{2}{\pi}} \frac{1}{2i} \sin(2x) \\
0 \\
0
\end{pmatrix}, \ldots, \begin{pmatrix}
\sqrt{\frac{2}{\pi}} \frac{1}{Ni} \sin(Nx) \\
0 \\
0
\end{pmatrix}, \\
\begin{pmatrix}
0 \\
\sqrt{\frac{2}{\pi}} \frac{1}{i} \sin(x) \\
0
\end{pmatrix}, \begin{pmatrix}
0 \\
\sqrt{\frac{2}{\pi}} \frac{1}{2i} \sin(2x) \\
0
\end{pmatrix}, \ldots, \begin{pmatrix}
0 \\
\sqrt{\frac{2}{\pi}} \frac{1}{Ni} \sin(Nx) \\
0
\end{pmatrix}, \\
\begin{pmatrix}
0 \\
0 \\
\sqrt{\frac{2}{\pi}} \frac{1}{i} \sin(x)
\end{pmatrix}, \begin{pmatrix}
0 \\
0 \\
\sqrt{\frac{2}{\pi}} \frac{1}{2i} \sin(2x)
\end{pmatrix}, \ldots, \begin{pmatrix}
0 \\
0 \\
\sqrt{\frac{2}{\pi}} \frac{1}{Ni} \sin(Nx)
\end{pmatrix}
\right\}.
\tag{2.1}
\end{align*}
\]

Based on the defined basis, we may approximate \( u, v, y, w \) by

\[
\begin{align*}
u^N(x, t) &= \sum_{i=1}^{N} \sqrt{\frac{2}{\pi}} \frac{1}{i} u_i(t) \sin(ix) \\
v^N(x, t) &= \sum_{i=1}^{N} \sqrt{\frac{2}{\pi}} \frac{1}{2i} v_i(t) \sin(ix) \\
y^N(x, t) &= \sum_{i=1}^{N} \sqrt{\frac{2}{\pi}} \frac{1}{3i} y_i(t) \sin(ix) \\
w^N(x, t) &= \sum_{i=1}^{N} \sqrt{\frac{2}{\pi}} \frac{1}{4i} w_i(t) \sin(ix)
\end{align*}
\tag{2.2}
\]

Therefore, \( Z_N \) and \( \dot{Z}_N \) could be written out as
\[
Z_N = \begin{bmatrix} u^N(x, t) \\ v^N(x, t) \\ y^N(x, t) \\ w^N(x, t) \end{bmatrix} = \begin{bmatrix} \sum_{i=1}^N \sqrt{\frac{2}{\pi}} u_i(t) \sin(ix) \\ \sum_{i=1}^N \sqrt{\frac{2}{\pi}} v_i(t) \sin(ix) \\ \sum_{i=1}^N \sqrt{\frac{2}{\pi}} y_i(t) \sin(ix) \\ \sum_{i=1}^N \sqrt{\frac{2}{\pi}} w_i(t) \sin(ix) \end{bmatrix},
\]

(2.3)

\[
\dot{Z}_N = \frac{d}{dt} Z_N = \begin{bmatrix} \dot{u}^N(x, t) \\ \dot{v}^N(x, t) \\ \dot{y}^N(x, t) \\ \dot{w}^N(x, t) \end{bmatrix} = \begin{bmatrix} \sum_{i=1}^N \sqrt{\frac{2}{\pi}} \dot{u}_i(t) \sin(ix) \\ \sum_{i=1}^N \sqrt{\frac{2}{\pi}} \dot{v}_i(t) \sin(ix) \\ \sum_{i=1}^N \sqrt{\frac{2}{\pi}} \dot{y}_i(t) \sin(ix) \\ \sum_{i=1}^N \sqrt{\frac{2}{\pi}} \dot{w}_i(t) \sin(ix) \end{bmatrix}.
\]

(2.4)

This \(Z_N\) satisfies equation

\[
\dot{Z}_N = AZ_N
\]

(2.5)

on the finite dimensional space \(H_N\) with

\[
AZ_N = \begin{bmatrix} 0 & I & 0 & 0 \\ -aD^4 - k(x) & -k(x)\delta & k(x) & 0 \\ 0 & 0 & I & 0 \\ k(x) & 0 & -aD^4 - k(x) & -k(x)\delta \end{bmatrix}
\]

\[
= \begin{bmatrix} \sum_{i=1}^N \sqrt{\frac{2}{\pi}} v_i(t) \sin(ix) \\ \sum_{i=1}^N \sqrt{\frac{2}{\pi}} y_i(t) \sin(ix) \\ \sum_{i=1}^N \sqrt{\frac{2}{\pi}} w_i(t) \sin(ix) \end{bmatrix},
\]

(2.6)

where,

\[
p(x, t) = -a \sum_{i=1}^N \sqrt{\frac{2}{\pi}} i^2 u_i(t) \sin(ix) - k(x) \sum_{i=1}^N \sqrt{\frac{2}{\pi}} i^2 u_i(t) \sin(ix) - k(x)\delta \sum_{i=1}^N \sqrt{\frac{2}{\pi}} v_i(t) \sin(ix)
\]

\[
+k(x) \sum_{i=1}^N \sqrt{\frac{2}{\pi}} \frac{1}{i^2} y_i(t) \sin(ix),
\]

and

\[
g(x, t) = -a \sum_{i=1}^N \sqrt{\frac{2}{\pi}} i^2 y_i(t) \sin(ix) - k(x) \sum_{i=1}^N \sqrt{\frac{2}{\pi}} i^2 y_i(t) \sin(ix) - k(x)\delta \sum_{i=1}^N \sqrt{\frac{2}{\pi}} w_i(t) \sin(ix)
\]

\[
+k(x) \sum_{i=1}^N \sqrt{\frac{2}{\pi}} \frac{1}{i^2} u_i(t) \sin(ix).
\]
2.2 The Operator $A_N$

We take the inner product of each side of $\hat{Z}_N = AZ_N$ with every element in the basis of the Hilbert Space $H_N$.

First, the inner product with the elements in the first line of (2.1),

$$\left[ \begin{array}{c} \sqrt{\frac{1}{\pi}} \sin(jx) \\ 0 \\ 0 \\ 0 \end{array} \right], \quad j = 1, 2, \ldots, N,$$

denoted by $c_{ij}^j$, is

$$\langle \hat{Z}_N, c_{ij}^j \rangle_{H_N} = \langle AZ_N, c_{ij}^j \rangle_{H_N}. \quad (2.7)$$

The left hand side of (2.7) is

$$\text{LHS} = \left\langle \left[ \begin{array}{c} \sum_{i=1}^{N} \sqrt{\frac{1}{\pi}} u_i(t) \sin(ix) \\ \sum_{i=1}^{N} \sqrt{\frac{1}{2}} v_i(t) \sin(ix) \\ \sum_{i=1}^{N} \sqrt{\frac{1}{2}} \dot{v}_i(t) \sin(ix) \\ \sum_{i=1}^{N} \sqrt{\frac{1}{2}} \ddot{v}_i(t) \sin(ix) \end{array} \right], \left[ \begin{array}{c} \sqrt{\frac{1}{\pi}} \sin(jx) \\ 0 \\ 0 \\ 0 \end{array} \right]\right\rangle_{H_N}, \quad j = 1, 2, \ldots N \quad (2.8)$$

$$= a \langle -\sum_{i=1}^{N} \sqrt{\frac{1}{\pi}} u_i(t) \sin(ix), -\sqrt{\frac{2}{\pi}} \sin(jx) \rangle_{L^2} + \langle k(x) \sum_{i=1}^{N} \sqrt{\frac{1}{2}} (\dot{v}_i(t) - \ddot{v}_i(t)) \sin(ix), \sqrt{\frac{2}{\pi}} \sin(jx) \rangle_{L^2} \quad (2.9)$$

$$= a \langle \dot{u}_j(t) + \sum_{i=1}^{N} \frac{1}{j} (\dot{v}_i(t) - \ddot{v}_i(t)), k(x) \sqrt{\frac{2}{\pi}} \sin(ix), \sqrt{\frac{2}{\pi}} \sin(jx) \rangle_{L^2}$$

Here, we use $d_{ij}$ to denote $\langle k(x) \frac{1}{j} \sqrt{\frac{2}{\pi}} \sin(ix), \sqrt{\frac{2}{\pi}} \sin(jx) \rangle_{L^2}$.

The right hand side of (2.7) is

$$\text{RHS} = \left\langle \left[ \begin{array}{c} \sum_{i=1}^{N} \sqrt{\frac{1}{\pi}} u_i(t) \sin(ix) \\ \sum_{i=1}^{N} \sqrt{\frac{1}{2}} v_i(t) \sin(ix) \\ \sum_{i=1}^{N} \sqrt{\frac{1}{2}} \dot{v}_i(t) \sin(ix) \\ \sum_{i=1}^{N} \sqrt{\frac{1}{2}} \ddot{v}_i(t) \sin(ix) \end{array} \right], \left[ \begin{array}{c} \sqrt{\frac{1}{\pi}} \sin(jx) \\ 0 \\ 0 \\ 0 \end{array} \right]\right\rangle_{H_N}, \quad j = 1, 2, \ldots N \quad (2.10)$$

$$= \langle -\sum_{i=1}^{N} \sqrt{\frac{1}{\pi}} u_i(t) \sin(ix), -\sqrt{\frac{2}{\pi}} \sin(jx) \rangle_{L^2} + \langle k(x) \sum_{i=1}^{N} \sqrt{\frac{1}{2}} (v_i(t) - w_i(t)) \sin(ix), \sqrt{\frac{2}{\pi}} \sin(jx) \rangle_{L^2} \quad (2.11)$$

$$= a_j^2 \dot{v}_j(t) + \sum_{i=1}^{N} \dot{w}_i(t) d_{ij}.$$ 

Therefore, we obtain from (2.9) and (2.11) that
\[ a \ddot{u}_j(t) + \sum_{i=1}^{N} (\ddot{u}_i(t) - \ddot{y}_i(t)) d_{ij} = a j^2 v_j(t) + \sum_{i=1}^{N} i^2 (v_i(t) - u_i(t)) d_{ij} . \]  

(2.12)

The inner product with the elements in the second line of (2.1),

\[
\begin{bmatrix}
0 \\
\sqrt{\frac{2}{\pi}} \sin(jx) \\
0 \\
\end{bmatrix}, \quad j = 1, 2, ..., N,
\]

by \( e^i \), is

\[
\langle \dot{Z}_N, e^i \rangle_{H^N} = \langle A Z_N, e^i \rangle_{H^N} .
\]  

(2.13)

Then, we have

\[
LHS = \left\langle \begin{bmatrix}
\sum_{i=1}^{N} \sqrt{\frac{2}{\pi}} \hat{u}_i(t) \sin(ix) \\
\sum_{i=1}^{N} \sqrt{\frac{2}{\pi}} \hat{v}_i(t) \sin(ix) \\
\sum_{i=1}^{N} \sqrt{\frac{2}{\pi}} \hat{y}_i(t) \sin(ix) \\
\sum_{i=1}^{N} \sqrt{\frac{2}{\pi}} \hat{w}_i(t) \sin(ix)
\end{bmatrix}, \begin{bmatrix}
0 \\
\sqrt{\frac{2}{\pi}} \sin(jx) \\
0 \\
\end{bmatrix} \right\rangle_{H^N}, \quad j = 1, 2, ...N
\]  

(2.14)

\[
= \left\langle \sum_{i=1}^{N} \sqrt{\frac{2}{\pi}} \hat{v}_i(t) \sin(ix), \sqrt{\frac{2}{\pi}} \sin(jx) \right\rangle_{L^2}
\]  

(2.15)

and

\[
RHS = \left\langle \begin{bmatrix}
\sum_{i=1}^{N} \sqrt{\frac{2}{\pi}} w_i(t) \sin(ix) \\
p(x,t) \\
qu(x,t)
\end{bmatrix}, \begin{bmatrix}
0 \\
\sqrt{\frac{2}{\pi}} \sin(jx) \\
0 \\
\end{bmatrix} \right\rangle_{H^N}, \quad j = 1, 2, ...N
\]  

(2.16)

\[
= \left\langle p(x,t), \sqrt{\frac{2}{\pi}} \sin(jx) \right\rangle_{L^2}
\]  

(2.17)

Therefore, we obtain from (2.15) and (2.17) that
\[ \frac{\dot{v}_i(t)}{t^2} = -a j^2 u_j(t) - \sum_{i=1}^{N} j^2 (u_i(t) + \delta v_i(t) - \frac{1}{t^2} y_i(t))d_{ij}. \] (2.18)

In the same way, the inner products with the elements \( e_j^Y = \begin{bmatrix} 0 \\ 0 \\ \sqrt{\frac{2}{\pi}} \frac{1}{j^2} \sin(jx) \\ 0 \end{bmatrix}, e_j^W = \begin{bmatrix} 0 \\ 0 \\ 0 \\ \sqrt{\frac{2}{\pi}} \sin(jx) \end{bmatrix}, j = 1, 2, ..., N \) are

\[ \langle Z_N, e_j^Y \rangle_{H_N} = \langle AZ_N, e_j^Y \rangle_{H_N}, \langle \dot{Z}_N, e_j^Y \rangle_{H_N} = \langle AZ_N, e_j^Y \rangle_{H_N}, \] (2.19)

which lead to

\[ a\dot{y}_j(t) + \sum_{i=1}^{N} (\dot{y}_i(t) - \dot{u}_i(t))d_{ij} = a j^2 w_j(t) + \sum_{i=1}^{N} j^2 (w_i(t) - v_i(t))d_{ij}. \] (2.20)

and

\[ \dot{w}_i(t) = -a j^2 y_j(t) - \sum_{i=1}^{N} j^2 j^2 (y_i(t) + \delta w_i(t) - \frac{1}{t^2} u_i(t))d_{ij}. \] (2.21)

From all above, we have

\[
\begin{align*}
da \dot{u}_j(t) + \sum_{i=1}^{N} (\dot{u}_i(t) - \dot{y}_i(t))d_{ij} &= a j^2 v_j(t) + \sum_{i=1}^{N} j^2 (v_i(t) - w_i(t))d_{ij} \\
\dot{v}_i(t) &= -a j^2 u_j(t) - \sum_{i=1}^{N} j^2 j^2 (u_i(t) + \delta v_i(t) - \frac{1}{t^2} y_i(t))d_{ij} \\
\dot{a} \dot{y}_j(t) + \sum_{i=1}^{N} (\dot{y}_i(t) - \dot{u}_i(t))d_{ij} &= a j^2 w_j(t) + \sum_{i=1}^{N} j^2 (w_i(t) - v_i(t))d_{ij} \\
\dot{a} \dot{w}_i(t) &= -a j^2 y_j(t) - \sum_{i=1}^{N} j^2 j^2 (y_i(t) + \delta w_i(t) - \frac{1}{t^2} u_i(t))d_{ij}
\end{align*}
\] (2.22)

The equations could be written as \( M_N \dot{Z}_N = \hat{A}_N Z_N \), where
\[
\begin{align*}
M_N &= \begin{bmatrix}
M_{11} & 0 & M_{13} & 0 \\
0 & I & 0 & 0 \\
M_{13} & 0 & M_{11} & 0 \\
0 & 0 & 0 & I
\end{bmatrix} \\
Z_N &= \begin{bmatrix}
\hat{u}_1 \\
\vdots \\
\hat{u}_N \\
\hat{v}_1 \\
\vdots \\
\hat{v}_N \\
\hat{y}_1 \\
\vdots \\
\hat{y}_N \\
\hat{w}_1 \\
\vdots \\
\hat{w}_N
\end{bmatrix} \\
\hat{A}_N &= \begin{bmatrix}
0 & \hat{A}_{12} & \hat{A}_{14} \\
\hat{A}_{21} & \hat{A}_{22} & \hat{A}_{23} \\
\hat{A}_{23} & 0 & \hat{A}_{21} & \hat{A}_{22}
\end{bmatrix} \\
Z_N &= \begin{bmatrix}
u_1 \\
v_2 \\
\vdots \\
y_1 \\
y_2 \\
\vdots \\
y_N \\
w_1 \\
w_2 \\
w_N
\end{bmatrix}
\end{align*}
\]

In the matrix \(M_N\),

\[
M_{11} = \begin{bmatrix}
a + d_{11} & d_{21} & \cdots & d_{N1} \\
d_{12} & a + d_{22} & \cdots & d_{N2} \\
\vdots & \vdots & \ddots & \vdots \\
d_{1N} & d_{2N} & \cdots & a + d_{NN}
\end{bmatrix}
\]

\[
M_{13} = \begin{bmatrix}
-d_{11} & -d_{21} & \cdots & -d_{N1} \\
d_{12} & -d_{22} & \cdots & -d_{N2} \\
\vdots & \vdots & \ddots & \vdots \\
d_{1N} & -d_{2N} & \cdots & -d_{NN}
\end{bmatrix}
\]

In matrix \(\hat{A}_N\),

\[
\hat{A}_{12} = \begin{bmatrix}
-a - d_{11} & -4d_{21} & \cdots & N^2d_{N1} \\
d_{12} & -4a - 4d_{22} & \cdots & N^2d_{N2} \\
\vdots & \vdots & \ddots & \vdots \\
d_{1N} & 4d_{2N} & \cdots & N^2a + N^2d_{NN}
\end{bmatrix}
\]

\[
\hat{A}_{14} = \begin{bmatrix}
-a - d_{11} & -4d_{21} & \cdots & -N^2d_{N1} \\
d_{12} & -4a - 16d_{22} & \cdots & -4N^2d_{N2} \\
\vdots & \vdots & \ddots & \vdots \\
d_{1N} & -4N^2d_{2N} & \cdots & -N^2a + N^4d_{NN}
\end{bmatrix}
\]

\[
\hat{A}_{21} = \begin{bmatrix}
-N^2d_{1N} & -4N^2d_{2N} & \cdots & -N^2a - N^4d_{NN}
\end{bmatrix}
\]

\[
\hat{A}_{22} = \begin{bmatrix}
-a - d_{11} & -4d_{21} & \cdots & -N^2d_N1 \\
d_{12} & -4a - 16d_{22} & \cdots & -4N^2d_{N2} \\
\vdots & \vdots & \ddots & \vdots \\
d_{1N} & -4N^2d_{2N} & \cdots & -N^2d_{NN}
\end{bmatrix}
\]

\[
\hat{A}_{23} = \begin{bmatrix}
-N^2d_{1N} & -4N^2d_{2N} & \cdots & -N^4d_{NN}
\end{bmatrix}
\]

\[
\begin{bmatrix}
d_{11} & d_{21} & \cdots & d_{N1} \\
d_{12} & 4d_{22} & \cdots & 4d_{N2} \\
\vdots & \vdots & \ddots & \vdots \\
N^2d_{1N} & N^2d_{2N} & \cdots & N^2d_{NN}
\end{bmatrix}
\]

Then this could be transformed to \(Z_N = \hat{M}_N^{-1}\hat{A}_NZ_N\). Therefore, \(A_N\) is actually \(\hat{M}_N^{-1}\hat{A}_N\).
3 Numerical Computation

We now have the operator $A_N = M^{-1}_N \hat{A}_N$. Therefore, if $N, k(x), a$ and $\delta$ are given, the eigenvalues of $A_N$ can be calculated by Matlab software.

3.1 Convergence

Our first experiment is for the case of one spring. Assume $N = 32$, $p = 0.1$, $s = 0.05$, $a = 1$, $\delta = 1$. We compute the eigenvalues and list the first 20 eigenvalues of lower frequencies. (i.e. their imaginary number part have the smallest values.)

<table>
<thead>
<tr>
<th>No.</th>
<th>Eigenvalue</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>-2054399138e-3 -1.000000861i</td>
</tr>
<tr>
<td>2</td>
<td>-2054399138e-3 +1.000000861i</td>
</tr>
<tr>
<td>3</td>
<td>-2019552097-3-1.000408144i</td>
</tr>
<tr>
<td>4</td>
<td>-2019552097-3+1.000408144i</td>
</tr>
<tr>
<td>5</td>
<td>-3197559943e-2-4.002476485i</td>
</tr>
<tr>
<td>6</td>
<td>-3197559943e-2+4.002476485i</td>
</tr>
<tr>
<td>7</td>
<td>-3147368278e-2-4.004060607i</td>
</tr>
<tr>
<td>8</td>
<td>-3147368278e-2+4.004060607i</td>
</tr>
<tr>
<td>9</td>
<td>-1548308465e-1-9.014673631i</td>
</tr>
<tr>
<td>10</td>
<td>-1548308465e-1+9.014673631i</td>
</tr>
<tr>
<td>11</td>
<td>-1526013867e-1-9.018076289i</td>
</tr>
<tr>
<td>12</td>
<td>-1526013867e-1+9.018076289i</td>
</tr>
<tr>
<td>13</td>
<td>-4610481934e-1-16.04791398i</td>
</tr>
<tr>
<td>14</td>
<td>-4610481934e-1+16.04791398i</td>
</tr>
<tr>
<td>15</td>
<td>-4550144921e-1-16.05359105i</td>
</tr>
<tr>
<td>16</td>
<td>-4550144921e-1+16.05359105i</td>
</tr>
<tr>
<td>17</td>
<td>-1047360576-25.11621176i</td>
</tr>
<tr>
<td>18</td>
<td>-1047360576+25.11621176i</td>
</tr>
<tr>
<td>19</td>
<td>-1035012688-25.12441592i</td>
</tr>
<tr>
<td>20</td>
<td>-1035012688+25.12441592i</td>
</tr>
</tbody>
</table>

Below is a graph with the eigenvalues in the given range,

We also list the first 10 eigenvalues of operator $A_N$ of lower frequencies with positive imaginary part, when $N = 32, 64, 128, 256$ and $512$. 
Figure 3.1: The eigenvalues in the given range, $N = 32, p = 0.1, s = 0.05, a = 1, \delta = 1$

<table>
<thead>
<tr>
<th>No.</th>
<th>N=32</th>
<th>N=64</th>
<th>N=128</th>
<th>N=256</th>
<th>N=512</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>-2054400e-3</td>
<td>-186201e-3</td>
<td>-179336e-3</td>
<td>-175816e-3</td>
<td>-174041e-3</td>
</tr>
<tr>
<td></td>
<td>+1.00000*i</td>
<td>+1.00000*i</td>
<td>+1.00000*i</td>
<td>+1.00000*i</td>
<td>+1.00000*i</td>
</tr>
<tr>
<td>2</td>
<td>-201955e-3</td>
<td>-183244e-3</td>
<td>-176563e-3</td>
<td>-173136e-3</td>
<td>-171408e-3</td>
</tr>
<tr>
<td></td>
<td>+1.00041*i</td>
<td>+1.00037*i</td>
<td>+1.00036*i</td>
<td>+1.00035*i</td>
<td>+1.00035*i</td>
</tr>
<tr>
<td>3</td>
<td>-319756e-2</td>
<td>-290175e-2</td>
<td>-279627e-2</td>
<td>-274217e-2</td>
<td>-271490e-2</td>
</tr>
<tr>
<td></td>
<td>+4.00248*i</td>
<td>+4.00224*i</td>
<td>+4.00216*i</td>
<td>+4.00212*i</td>
<td>+4.00210*i</td>
</tr>
<tr>
<td>4</td>
<td>-314737e-2</td>
<td>-285900e-2</td>
<td>-275612e-2</td>
<td>-270335e-2</td>
<td>-267674e-2</td>
</tr>
<tr>
<td></td>
<td>+4.00406*i</td>
<td>+4.00368*i</td>
<td>+4.00355*i</td>
<td>+4.00348*i</td>
<td>+4.00344*i</td>
</tr>
<tr>
<td>5</td>
<td>-154831e-1</td>
<td>-140781e-1</td>
<td>-135778e-1</td>
<td>-133211e-1</td>
<td>-131917e-1</td>
</tr>
<tr>
<td>6</td>
<td>-152601e-1</td>
<td>-138873e-1</td>
<td>-133982e-1</td>
<td>-131473e-1</td>
<td>-130208e-1</td>
</tr>
<tr>
<td>7</td>
<td>-461048e-1</td>
<td>-420237e-1</td>
<td>-405740e-1</td>
<td>-398301e-1</td>
<td>-394550e-1</td>
</tr>
<tr>
<td>8</td>
<td>-455014e-1</td>
<td>-415043e-1</td>
<td>-400842e-1</td>
<td>-393556e-1</td>
<td>-389881e-1</td>
</tr>
<tr>
<td></td>
<td>+16.0536*i</td>
<td>+16.0486*i</td>
<td>+16.0468*i</td>
<td>+16.0459*i</td>
<td>+16.0455*i</td>
</tr>
<tr>
<td>9</td>
<td>-104736</td>
<td>-957224e-1</td>
<td>-925332e-1</td>
<td>-908971e-1</td>
<td>-900718e-1</td>
</tr>
<tr>
<td></td>
<td>+25.1162*i</td>
<td>+25.1054*i</td>
<td>+25.1016*i</td>
<td>+25.0996*i</td>
<td>+25.0986*i</td>
</tr>
<tr>
<td>10</td>
<td>-103501</td>
<td>-946527e-1</td>
<td>-915221e-1</td>
<td>-899161e-1</td>
<td>-891060e-1</td>
</tr>
<tr>
<td></td>
<td>+25.1244*i</td>
<td>+25.1129*i</td>
<td>+25.1089*i</td>
<td>+25.1068*i</td>
<td>+25.1057*i</td>
</tr>
</tbody>
</table>

In this table, eigenvalues are ranked by the value of imaginary part. Look at row 1 and row 2, when dimension of the space increases from 32 to 64, 64 to 128, 128 to 256, 256 to 512, the changes of the real part and imaginary part of the eigenvalue becomes smaller and closer to zero. For example, when $N$ increases from
64 to 128, the real part increases by 0.006865 and the imaginary part’s change close to 0; when N increases from 128 to 256, the real part increases by 0.00352 and the imaginary part’s change is close to 0. Therefore, it could be claimed that the 1st eigenvalue is going to converge to a certain point when N goes to infinity.

It is observed that every row has the same properties as row 1 and row 2. Therefore, it could be claimed that the eigenvalues of the operator $A_N$ would converge to certain points when N approaches to infinity.
Below is the graphs of the convergence of the 1st and 2nd eigenvalues in the table.

Figure 3.2: Showing the convergence of the 1st eigenvalue, $p = 0.1, s = 0.05, a = 1, \delta = 1$

Figure 3.3: Showing the convergence of the 2nd eigenvalue, $p = 0.1, s = 0.05, a = 1, \delta = 1$
Next, we consider a case of two springs. Take $N = 64, p_1 = 0.1, p_2 = 2.1, s = 0.05, a = 1, \delta = 1$. We list the first 20 eigenvalues of lower frequencies.

<table>
<thead>
<tr>
<th>No.</th>
<th>Eigenvalue</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>-1.337119832e-1 + 1.000632777i</td>
</tr>
<tr>
<td>2</td>
<td>-1.337119832e-1 - 1.000632777i</td>
</tr>
<tr>
<td>3</td>
<td>-1.277976021e-1 + 1.025874325i</td>
</tr>
<tr>
<td>4</td>
<td>-1.277976021e-1 - 1.025874325i</td>
</tr>
<tr>
<td>5</td>
<td>-5.714003666e-1 + 4.043338560i</td>
</tr>
<tr>
<td>6</td>
<td>-5.714003666e-1 - 4.043338560i</td>
</tr>
<tr>
<td>7</td>
<td>-5.641546881e-1 + 4.071339054i</td>
</tr>
<tr>
<td>8</td>
<td>-5.641546881e-1 - 4.071339054i</td>
</tr>
<tr>
<td>9</td>
<td>-1.538191124e-1 + 9.044701408i</td>
</tr>
<tr>
<td>10</td>
<td>-1.538191124e-1 - 9.044701408i</td>
</tr>
<tr>
<td>11</td>
<td>-1.518614237e-1 + 9.017542542i</td>
</tr>
<tr>
<td>12</td>
<td>-1.518614237e-1 - 9.017542542i</td>
</tr>
<tr>
<td>13</td>
<td>-2.456095233 + 1.624894815i</td>
</tr>
<tr>
<td>14</td>
<td>-2.456095233 - 1.624894815i</td>
</tr>
<tr>
<td>15</td>
<td>-2.427389730 + 1.627890557i</td>
</tr>
<tr>
<td>16</td>
<td>-2.427389730 - 1.627890557i</td>
</tr>
<tr>
<td>17</td>
<td>-4.446751779 + 25.45874338i</td>
</tr>
<tr>
<td>18</td>
<td>-4.446751779 - 25.45874338i</td>
</tr>
<tr>
<td>19</td>
<td>-4.433719211 + 25.49363828i</td>
</tr>
<tr>
<td>20</td>
<td>-4.433719211 - 25.49363828i</td>
</tr>
</tbody>
</table>

Below is a graph with the eigenvalues in the given range,

![Graph](image_url)

Figure 3.4: The eigenvalues in the given range, $N = 64, p_1 = 0.1, p_2 = 2.1, s = 0.05, a = 1, \delta = 1$
We also list the first 10 eigenvalues of operator $A_N$ of lower frequencies with positive imaginary number, when $N=32, 64, 128, 256$ and $512$.

<table>
<thead>
<tr>
<th>No.</th>
<th>N=32</th>
<th>N=64</th>
<th>N=128</th>
<th>N=256</th>
<th>N=512</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>-1.198246e-1</td>
<td>-1.88330e-1</td>
<td>-1.81685e-1</td>
<td>-1.17832e-1</td>
<td>-1.176665e-1</td>
</tr>
<tr>
<td>2</td>
<td>-1.192727e-1</td>
<td>-1.83322e-1</td>
<td>-1.76995e-1</td>
<td>-1.173781e-1</td>
<td>-1.172212e-1</td>
</tr>
<tr>
<td></td>
<td>+1.03831i</td>
<td>+1.03645i</td>
<td>+1.03520i</td>
<td>+1.03457i</td>
<td>+1.03426i</td>
</tr>
<tr>
<td></td>
<td>+4.07312i</td>
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<td>+4.06712i</td>
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</tr>
<tr>
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<td>+4.11275i</td>
<td>+4.10875i</td>
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<td>+4.10581i</td>
</tr>
<tr>
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<tr>
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<tr>
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</tr>
<tr>
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<td>+16.6032i</td>
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<td>+16.5492i</td>
<td>+16.5387i</td>
<td>+16.5337i</td>
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<tr>
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</tr>
<tr>
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<td>+25.3288i</td>
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<td>+25.2967i</td>
<td>+25.2909i</td>
<td>+25.2880i</td>
</tr>
<tr>
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<td>-2.83030</td>
<td>-2.77724</td>
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</tr>
<tr>
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<td>+25.3535i</td>
<td>+25.3317i</td>
<td>+25.3191i</td>
<td>+25.3128i</td>
<td>+25.3097i</td>
</tr>
</tbody>
</table>

In this table, eigenvalues are ranked by the value of imaginary part. When dimension of the space increases from 32 to 64, 64 to 128, 128 to 256, 256 to 512, the changes of the real part and imaginary part of the eigenvalue becomes smaller and closer to zero. Therefore, it could be claimed that every eigenvalue would converge to certain points when $N$ goes to infinity.
Below is the graphs of the convergence of the 1st and 2nd eigenvalues in the table.

Figure 3.5: Showing the convergence of the 1st eigenvalue, \( p_1 = 0.1, p_2 = 2.1, s = 0.05, a = 1, \delta = 1 \)

Figure 3.6: Showing the convergence of the 2nd eigenvalue, \( p_1 = 0.1, p_2 = 2.1, s = 0.05, a = 1, \delta = 1 \)
3.2 Optimal Spring-Damper Location - Case of One Spring

As mentioned before, our goal is to find the optimal Spring-damper locations. At each location \( p \), \( r_0^N(p) \), the maximum real part of the eigenvalue, would determine the decay rate of the energy of the system. We need to find the maximum of \( |r_0^N(p)| \) over all \( p \).

Taking 100 or 300 mesh points evenly distributed on \([0, \pi]\) as the center of the Spring-damper location \( p \), and then we find out the value of \( |r_0^N(p)| \). Therefore, we would be able to observe the change of the decay rate with the change of \( p \).

The next two graphs are the value of \( |r_0^N(p)| \) at the mesh points. In the first graph, we take 300 mesh points and dimension \( N=128 \); In the second one, we take 100 mesh points and dimension \( N=256 \).

![Figure 3.7: The value of \( |r_0^N(p)| \) at the mesh points \( \frac{j\pi}{200}, \ j = 1, 2, ..., 200 \), \( N = 128 \), \( s = 0.05 \), \( a = 1 \), \( \delta = 1 \)](image-url)
Figure 3.8: The value of $|r_0^N(p)|$ at the mesh points $\frac{j \pi}{100}, \ j = 1, 2, ..., 100, \ N = 256, s = 0.05, a = 1, \delta = 1$
3.3 Optimal Spring-damper Locations - Case of Two Springs

Let $p_1, p_2$ be the centers of the Spring-dampers location. Taking 300 mesh points evenly distributed on $[0, \pi]$. We compute \(|r_0^N(\frac{p_1+p_2}{2})|\) for all admissible mesh points for $d = p_2 - p_1 = \frac{\pi}{30}, \frac{2\pi}{30}, \ldots, \pi$.

The following graphs are the values of \(|r_0^N(p)|\) at the mesh points and at each given $d$ value. In all these graphs, we take 300 mesh points and dimension $N=64$. We keep changing the distance of the centers of the two springs to get the graphs below.

Figure 3.9: \(|r_0^N(\frac{p_1+p_2}{2})|\) at the mesh points, $d = p_2 - p_1 = \frac{\pi}{30}, N = 64, s = 0.05, a = 1, \delta = 1$
Figure 3.10: $|r^N_0\left(\frac{p_1+p_2}{2}\right)|$ at the mesh points, $d = p_2 - p_1 = \frac{2\pi}{30}$, $N = 64$, $s = 0.05$, $\alpha = 1$, $\delta = 1$
Figure 3.11: $|r_0^N(\frac{p_1+p_2}{2})|$ at the mesh points, $d = p_2 - p_1 = \frac{3\pi}{30}$, $N = 64, s = 0.05, a = 1, \delta = 1$
Figure 3.12: $|r_0^N \left( \frac{p_1 + p_2}{2} \right)|$ at the mesh points, $d = p_2 - p_1 = \frac{4\pi}{30}$, $N = 64$, $s = 0.05$, $\alpha = 1$, $\delta = 1$
Figure 3.13: $|r_0^N \left( \frac{p_1 + p_2}{2} \right)|$ at the mesh points, $d = p_2 - p_1 = \frac{5\pi}{30}$, $N = 64$, $s = 0.05$, $a = 1$, $\delta = 1$
Figure 3.14: $|r_0^N(p_1+p_2)/2)$ | at the mesh points, $d = p_2 - p_1 = \frac{6\pi}{50}$, $N = 64$, $s = 0.05$, $a = 1$, $\delta = 1$
Figure 3.15: $|r_0^N(p_1 + p_2)/2|$ at the mesh points, $d = p_2 - p_1 = \frac{7\pi}{30}$, $N = 64$, $s = 0.05$, $a = 1$, $\delta = 1$.
Figure 3.16: $|r^N_0\left(\frac{p_1+p_2}{2}\right)|$ at the mesh points, $d = p_2 - p_1 = \frac{2\pi}{30}$, $N = 64$, $s = 0.05$, $a = 1$, $\delta = 1$
Figure 3.17: \( |r_d^N(\frac{p_1+p_2}{2})| \) at the mesh points, \( d = p_2 - p_1 = \frac{9\pi}{50} \), \( N = 64, s = 0.05, a = 1, \delta = 1 \)
Figure 3.18: $|r_0^N(p_1+p_2)|$ at the mesh points, $d = p_2 - p_1 = \frac{4\pi}{30}$, $N = 64$, $s = 0.05$, $a = 1$, $\delta = 1$
Since the maximum value of $|r_0^N(p)|$ decreases as $d = p_2 - p_1$ increases from $\frac{10\pi}{30}$ to $\pi$, we don’t include them graphs here.

We enlarge the figure 3.13 to identify the maximum $|r_0^N\left(\frac{p_1 + p_2}{2}\right)|$.

Figure 3.19: $|r_0^N\left(\frac{p_1 + p_2}{2}\right)|$ at the mesh points,$d = p_2 - p_1 = \frac{5\pi}{30}$, $N = 64$, $s = 0.05$, $a = 1, \delta = 1$. 

32
4 Conclusions

For the case of one spring, from the graphs above, we see that the $|r^N_0(p)|$ has the largest value 0.01601 when $p$ is close to $\frac{126\pi}{300}$ and $\frac{174\pi}{300}$. Therefore, the optimal location should be $p=\frac{126\pi}{300}$ or $p=\frac{174\pi}{300}$.

For the case of two springs, from the graphs above, we see that $|r^N_0(p)|$ has the largest value 0.03324 when $p_2 - p_1 = \frac{5\pi}{30}$ and $\frac{p_1 + p_2}{2} = \frac{\pi}{2}$. Therefore, the optimal locations should be close to $\frac{5\pi}{12}$ and $\frac{7\pi}{12}$.
References

