Living with the Labeling Disease for 25 years

Joe Gallian
University of Minnesota Duluth
My background

• PhD in finite groups 1971
• Did not know definition of graph when I stated at UMD 1972
• Backed into graph theory through Cayley graphs—gave problems to my summer research students
• I looked at many papers on Hamiltonian circuits in Cayley digraphs looking for problems for students
I enjoy expository writing and contributing to the research literature so writing surveys is a good fit for me.

Published two surveys on Hamiltonian circuits in Cayley digraphs.

Journal editors are eager to receive surveys.
• My surveys are read and cited far more than my other papers
• First Cayley digraph survey is cited on Google scholar 80 times
• Second Cayley digraph survey is cited 74 times
First encounter with graph labeling

• First heard of graph labeling from Ron Graham—harmonious labeling
• First problem I worked on was graceful labeling prisms $C_n \times P_2$ with Frucht about 1986
• First two graphs I ever labeled were $C_3 \times P_2$ and $C_4 \times P_2$
• Third was
6th INTERNATIONAL WORKSHOP ON GRAPH LABELINGS

IWOGL 2010  October 20–22, 2010
University of Minnesota  Duluth

IWOGL 2010 will cover all aspects of graph labelings. There will be four keynote lectures, several invited talks, contributed talks, and hopefully enough time for discussions and collaborative research.

Keynote speakers:
S. Arumugam, Kalasalingam University, India
Joseph Gallian, University of Minnesota Duluth, U.S.A.
Uwe Leck, University of Wisconsin-Superior, U.S.A.
Alexander Rosa, McMaster University, Canada

Library Rotunda 4th floor
Keynote Lectures daily 9AM and Th 3 PM
Invited & Contributed Talks 10:30–12, 2–6

Organizing Committee:
Dailbor Froncek (Chair) UMD
Sergei Bezrukov, Steven Rosenberg UWS

email: iwogl@d.umn.edu
http://www.d.umn.edu/math/iwogl/
Graceful Labeling

5 edges use vertex labels 0-5
Create edge labels 1-5 by subtraction
Harmonious Labeling

5 edges use vertex labels 0-4
Create edge labels 1-5 by addition modulo 5
• First gave labeling problems to students in 1986—Jungreis & Reid--Graceful tree conjecture
My first labeling survey paper in JGT 1989
A Survey: Recent Results, Conjectures, and Open Problems in Labeling Graphs

Joseph A. Gallian
DEPARTMENT OF MATHEMATICS
AND STATISTICS
UNIVERSITY OF MINNESOTA, DULUTH
DULUTH, MINNESOTA

ABSTRACT

In this paper we organize and summarize much of the work done on graceful and harmonious labelings of graphs. Many open problems and conjectures are included.

1. INTRODUCTION

Interest in graph labeling problems began in the mid-1960s with a conjecture of G. Ringel [64] and a paper by A. Rosa [66]. In the intervening two decades, well over 150 papers on this topic have appeared. The so-called Ringel–Kotzig conjecture that all trees are graceful has been the focus of many of these (see [6,11,46,78]). Despite the large number of papers, there are relatively few general results or methods on graph labelings. Indeed, most of the results focus on particular classes of graphs and utilize ad hoc methods. Frequently, the same classes have been done by several authors. Labeled graphs serve as useful models for a broad range of applications such as: coding theory, x-ray crystallography, radar, astronomy, circuit design, and communication network addressing (see [12] and [13] for details). In this paper we organize and summarize much of the work done to date, and offer a plethora of open problems and conjectures. Earlier surveys include [6,11,46,78].
JGT survey

- 14 pages—graceful and harmonious only
- 78 references
- In introduction I mention 150 papers written
- JGT survey cited on Google scholar 61 times
Second labeling survey—Graph labeling zoo

• Discrete Applied Mathematics 1994--17 pages
• Updated first survey on graceful and harmonious and added variations such as cordial, sequential, etc.
• 74 references
• Mention in introduction “over 200 papers”
• Cited 21 times on Google scholar
A guide to the graph labeling zoo

Joseph A. Gallian

Department of Mathematics and Statistics, University of Minnesota, Duluth, MN 55812, USA

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Abstract

In this paper we survey many of the variations of graceful and harmonious labeling methods that have been introduced and summarize much of what is known about each kind.

1. Introduction

A vertex labeling, valuation or numbering of a graph G is an assignment f of labels to the vertices of G that induces for each edge xy a label depending on the vertex labels f(x) and f(y). Most graph labeling methods trace their origin to one introduced by Rosa [Ga66] in 1967, or one given by Graham and Sloane [Ga36] in 1980. Rosa [Ga66] called a function f a \( \beta \)-valuation of a graph G with \( q \) edges if f is an injection from the vertices of G to the set \( \{0, 1, \ldots, q\} \) such that, when each edge \( xy \) is assigned the label \( |f(x) - f(y)| \), the resulting edge labels are distinct. (Golomb [Ga32] subsequently called such labelings graceful and this term is now the popular one.) Rosa introduced \( \beta \)-valuations as well as a number of other valuations as tools for decomposing the complete graph into isomorphic subgraphs. In particular, \( \beta \)-valuations originated as a means of attacking the conjecture of Ringel [Ga64] that \( K_{2n+1} \) can be decomposed into \( 2n + 1 \) subgraphs that are all isomorphic to a given tree with \( n \) edges. Harmonious graphs naturally arose in the study by Graham and Sloane [Ga36] of modular versions of additive bases problems stemming from error-correcting codes. They defined a graph G with \( q \) edges to be harmonious if there is an injection f from the vertices of G to the group of integers modulo q such that, when each edge \( xy \) is assigned the label \( f(x) + f(y) \pmod{q} \), the resulting edge labels are distinct. When G is a tree, exactly one label may be used on two vertices.

In the intervening years, close to 200 papers have spawned a bewildering array of graph labeling methods. Despite the unabated procession of papers, there are few

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\( ^1 \)References whose number is preceded by Ga refer to an earlier survey by the author [25]. References denoted with a plain number are at the end of this paper.

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Dynamic survey

• Submitted September 1, 1996
• Accepted: November 14, 1997 (could not find referee)
• Second edition 1998 (43 pages)
• Third edition 1999 (52 pages)
• Four edition 2000 (?) pages
• Fifth edition 2000 (58 pages)
• Sixth edition 2001 (74 pages)
• Seventh 2002 (106 pages)
• Eight edition 2003 (147 pages)
• Ninth edition 2005 (155 pages)
• Tenth edition 2006 (196 pages) [794] references
• Eleventh edition 2008 (197 pages—EJC font) [865] references
• Twelfth edition 2009 219 pages [1013] references
• IWOGL edition 2010 233 pages [1197] references
Cited 404 times on Google Scholar
My graph labeling papers pile
Data mining
Most cited papers in labeling

- A. Rosa, On certain valuations of the vertices of a graph (285)
- Graham and Sloane, Additive bases and harmonious graphs (124)
- R. Stanley, Linear homo. Dioph. Eq. and magic labelings (122)
- Kotzig and Rosa, Magic valuations of finite graphs (120)
- Magic Graph book by Wallis (99)
- Enomoto, Llado, Nakamigawa, Ringel, Super edge-magic ... (66)
- Figueroa-Centenoa, Ichishima, Muntaner-Batle, Place of super edge-magic ...(53)
- Grace, Sequential labelings ... (53)
- Doob, Characterizations of regular magic graphs (48)
- S P Lo, On edge-graceful labelings (44)
- MacDougall, Miller, Wallis, Vertex magic total... (44)
- Bloom, Chronology of the Ringel-Kotzig Conjecture... (44)
Most papers by authors cited in survey

- S.M. Lee 135
- Baca 82
- Miller 67
- Sethuraman 30
- Slamin 27
- Hegde 32
- Ryan 25
- Acharaya 22
- Shiu 21
- Youssef 21
- Shetty 19
- Wallis 19
- Baskoro 19
- MacDougall 18

Seoud 18
Singh 16
Ng 16
Vilfred 16
Barrientos 15
Kotzig 14
Cahit 14
El-Zanati 12
Koh 12
Bu 11
Rogers 11
Figueroa-Centeno, Ichishima,
Muntaner-Batle 11
Gray, Kathiresan, Kwong, Seah,
Selvaraju 10
Labeling Time Line

- 1966  Magic: Sedlacek
- 1967  Graceful: Rosa
- 1967  Super magic: Steward
- 1970  Edge-magic total: Kotzig and Rosa
- 1980  Prime labeling: Entringer
- 1980  Harmonious: Graham and Sloane
- 1981  Sequential (strongly c-harmonious): Grace; Chang, Hsu, Rogers
- 1981  Elegant: Chang, Hsu, Rogers
- 1982  $k$-graceful: Slater; Maheo and Thuillier
- 1983  Magic labelings of Types $(a,b,c)$: Lih
- 1985  Edge-graceful: Lo
Labeling Time Line

- 1987 Cordial: Cahit
- 1990 \((k,d)\) – arithmetic, \((k,d)\) – indexable: Acharya – Hegde
- 1990 Antimagic: Hartsfield and Ringel
- 1990 \(k\)-equitable: Cahit
- 1990 Sum graphs: Harary
- 1990 Mod sum graphs: Boland, Lasker, Turner, Domke
- 1991 Skolem-graceful: Lee
- 1991 Odd-graceful: Gnanajothi
- 1991 Felicitous: Choo
- 1993 \((a,d)\)-antimagic: Bodendiek and Walther
- 1994 Integral sum graphs: Harary
- 1998 Super edge-magic: Enomoto, Llado, Nakamigana
- 1999 Vertex-magic total: MacDongall, Miller, Slamin, Wallis
- 2000 \((a,d)\) -Vertex-antimagic total labeling: Baca, Bertault, MacDougall, Miller, Simanjuntak, Slamin
- 2001 Radio labeling: Chartand, Erwin, Zhang, Harary
Most active labeling countries

My guess--Not sure of order
U.S., India, China, Australia, Indonesia, Slovakia
Some partially done problems I would like to see done

- $C_m \times C_n$ finish graceful and harmonious (page 14)
- $C_m \times P_n$ finish graceful and harmonious (page 14)
- Cycles with a $P_k$ cord harmonious ($k > 2$); graceful case done (page 10)
- $C_m \times C_n$ finish super edge magic (page 72) — $m$ and $n$ even case done
- $P_m \times P_n$ finish showing which have prime labelings (p. 135)
More mop up open problems

All lobsters graceful/harmonious (probably very hard) caterpillars are done in both cases

\( K_{4}^{(m)} (m > 3) \) conjectured graceful –done up to 33 (p. 16) (harmonious case done)

\( P_{n}^{k} \) harmonious open for some even \( k \) (odd case true)(p. 20)

\( P_{n}^{k} \) graceful open for \( k > 2 \) (p. 20)
Let $G$ be a graph with $q$ edges and $H$ a finite Abelian group (under addition) of order $q$. Define $G$ to be $H$-harmonious if there is an injection $f$ from the vertices of $G$ to $H$ such that when each edge $xy$ is assigned the label $f(x) + f(y)$ the resulting edge labels are distinct. When $G$ is a tree, one label may be used on exactly two vertices.

Only case done is cycles.
Beals, Gallian, Headley, and Jungreis have shown that if $H$ is a finite Abelian group of order $n > 1$ then $C_n$ is $H$-harmonious if and only if $H$ has a non-cyclic or trivial Sylow 2-subgroup and $H$ is not of the form $Z_2 \times Z_2 \times \cdots \times Z_2$.

Thus, for example, $C_{12}$ is not $Z_{12}$-harmonious but is $Z_2 \times Z_2 \times Z_3$-harmonious.
Every tree is graceful harmonious $k$-graceful for some $k$ odd-graceful odd-harmonious triangular graceful edge-magic total labelings antimagic (except $P_2$) $(a, 1)$-edge-antimagic total labeling prime
Cahit proved that every tree is cordial; not all trees have an $\alpha$-labeling; all trees are indexable; not all trees are elegant
New topic to 2010 survey
Ranking Numbers of Graphs

Arose in late 1980s in connection with very large scale integration (VLST) layout designs and parallel processing.
A *ranking* of a graph is a labeling of the vertices with positive integers such that every path between vertices of the same label contains a vertex of greater label. The *rank number* of a graph is the smallest possible number of labels in a ranking.
Known results

- (1996) Lasker and Pillone proved that determining $k$-rankings is NP-complete. It is HP-hard even for bipartite graphs.
- (1997) Wang found rank number of paths and joins.
- (1998) Bodlaender, Deogun, Jansen, Kloks, Kratsch, Muller, Tuza found the rank number of paths.
- (2004) Dere found the rank numbers of stars, cycles, wheels, and complete $k$-partite graphs.
• In 2009 Novotny, Ortiz, and Narayan determined the rank number of $P_n^2$ and asked about $P_n^k$.

• (2009) Alpert uses clever recursive methods to find rank numbers of $P_n^k$, $C_n^k$, prisms, Mobius ladders, $K_s \times P_n$, and $P_3 \times P_n$.

• She also found bounds for rank numbers of general grid graphs $P_m \times P_n$. 
About the same time as Alpert and independently, Chang, Kuo, and Lin determined the rank numbers of $P_n^k, C_n^k$, $P_2 \times P_n$, $P_2 \times C_n$.

Chang et al. also determined the rank numbers of caterpillars.
Open ranking number problems

Complete results on grid graphs

$P_m \times C_n$ (probably hard)

$C_m \times C_n$ (probably very hard)
How to help me with survey

• Send me preprints, corrections, typos, updates in citation information
• Current backlog—at least 50 papers
• Send me **useful** summaries of your papers.
Minimally useful

We provide a method for constructing a larger graceful tree by combining smaller ones that need not be identical. Our construction generalizes earlier methods for combining smaller trees.
Very useful

• Extended abstracts
Mavronicolas and Michael say that trees \( (T_1, \theta_1, w_1) \) and \( (T_2, \theta_2, w_2) \) with roots \( w_1 \) and \( w_2 \) and \( |V(T_1)| = |V(T_2)| \) are gracefully consistent if either they are identical or they have \( \alpha \)-labelings with the same boundary value and \( \theta_1(w_1) = \theta_2(w_2) \).

Mavronicolas and Michael use this concept to show that a number of known constructions of new graceful trees using several identical copies of a given graceful rooted tree can be extended to the case where the copies are replaced by a set of pairwise gracefully consistent trees. Let \( (T, \theta, w), (T_0, \theta_0, w_0) \) are gracefully labeled trees rooted at \( w \) and \( w_0 \) respectively.

1. Suppose \( \theta(w) = |E(T)| \); then the garland construction due to Koh, Rogers, and Tan gracefully labels the tree consisting of \( h \) copies of \( (T, w) \) with their roots connected to a new vertex \( r \).
2. Suppose \( \theta(w) = |E(T)| \) and assume also that whenever \( uw \in E(T) \) and \( \theta(u) \neq 0 \), then \( vw \in E(T) \) where \( \theta(u) + \theta(v) = |E(T)| \). The attachment construction of Koh, Tan and Rogers gracefully labels the tree formed by identifying the roots of \( h \) copies of \( (T, w) \).
3. The \( \Delta \)-construction given by Koh, Tan and Rogers gracefully labels the tree formed by merging each vertex of \( (T_0, w_0) \) with the root of a distinct copy of \( (T, w) \).
4. Let \( \theta_0(w_0) = |E(T_0)| \); let \( N \) be the set of neighbors of \( w_0 \) and let \( x \) be the vertex of \( T \) at even distance from \( w \) with \( \theta(x) = 0 \) or \( \theta(x) = |E(T)| \). The \( \Delta_{+1} \)-construction Burzio and Ferrarese gracefully labels the tree formed by merging each non-root vertex of \( T_0 \) with the root of a distinct copy of \( (T, w) \) so that for each \( v \in N \) the edge \( vw_0 \) is replaced with a new edge \( xw_0 \) (where \( x \) is in the corresponding copy of \( T \)).

Mavronicolas and Michael use a “relabeling function” for any \( (T, \theta, w) \) and any triple of integers \( (c, e, o) \):

\[
R^{(T, \theta, w)}_{(c, e, o)}(v) = c(\theta(v) + e), \text{ if } v \text{ is an even distance from } w;
\]

\[
R^{(T, \theta, w)}_{(c, e, o)}(v) = c(\theta(v) + o), \text{ if } v \text{ is an odd distance from } w.
\]

Their principal tool is their Substitution Theorem which states that if \( (T_i, \theta_i, w_i) \) are gracefully consistent \( (i = 1, 2) \), then for any \( (c, e, o) \) the labelings \( R^{(T_i, \theta_i, w_i)}_{(c, e, o)} \) give the same labels to the \( w_i \), the same label sets to the two vertex sets, and the same induced edge label sets to the two edge sets. The four constructions defined above are then shown to be adaptable to the case when a set of copies of \( (T, \theta, w) \) is replaced by a set of pairwise gracefully consistent trees.
Thank you for your contributions to graph labeling

Continue to spread the “disease”