1. Let $A = \begin{bmatrix} 0 & 1 & 3 \\ 0 & 0 & -2 \\ 0 & 0 & 0 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 3 & 0 \\ 1 & 2 & 0 \\ 0 & 0 & 4 \end{bmatrix}$. Compute

(a) (4 pts) AB

$$\begin{bmatrix} 1 & 2 & 12 \\ 0 & 0 & -8 \\ 0 & 0 & 0 \end{bmatrix}$$

(b) (6 pts) B^{-1}

$$\left[\begin{array}{ccc|ccc} 1 & 3 & 0 & 1 & 0 & 0 \\ 1 & 2 & 0 & 0 & 1 & 0 \\ 0 & 0 & 4 & 0 & 0 & 1 \end{array}\right] \longrightarrow \left[\begin{array}{ccc|ccc} 1 & 3 & 0 & 1 & 0 & 0 \\ 0 & -1 & 0 & -1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & \frac{1}{4} \end{array}\right] \longrightarrow \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -2 & 3 & 0 \\ 0 & 1 & 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & 0 & 0 & \frac{1}{4} \end{array}\right]$$

(c) (4 pts) Check your answer to (b) by multiplying B times B^{-1}.

$$\begin{bmatrix} 1 & 3 & 0 \\ 1 & 2 & 0 \\ 0 & 0 & 4 \end{bmatrix} \begin{bmatrix} -2 & 3 & 0 \\ 1 & -1 & 0 \\ 0 & 0 & \frac{1}{4} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \checkmark$$

2. (6 pts) Evaluate the following determinant. Show your work.

$$\begin{vmatrix} 0 & 3 & -1 & 2 \\ 1 & 0 & 3 & 1 \\ 1 & 0 & -1 & -1 \\ 0 & 0 & -2 & 3 \end{vmatrix}$$

2nd col:

$$= (-3) \begin{vmatrix} 1 & 3 & 1 \\ 1 & -1 & -1 \\ 0 & -2 & 3 \end{vmatrix} + 0 + 0 + 0 = (-3)\left(1\begin{vmatrix} -1 & -1 \\ -2 & 3 \end{vmatrix} - 1\begin{vmatrix} 3 & 1 \\ -2 & 3 \end{vmatrix} + 0\right)$$

$$= (-3)(-5 - 11) = (-3)(-16)$$

$$= -3(-16) = 48$$

3. (8 pts) Solve the following linear system USING GAUSSIAN ELIMINATION (row reduction to echelon or reduced echelon form). Leave your answers as exact fractions - not calculator approximations.

$$\begin{bmatrix} 1 & 3 \\ 1 & -3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$$

$$\left[\begin{array}{cc|c} 1 & 3 & 1 \\ 1 & -3 & 3 \end{array}\right] \rightarrow \begin{bmatrix} 1 & 3 & 1 \\ 0 & -6 & 2 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 3 & 1 \\ 0 & 1 & -\frac{1}{3} \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & -\frac{1}{3} \end{bmatrix}, \text{ so } x_1 = 2, \; x_2 = -\frac{1}{3}$$

4. (5 pts) Circle any of the following sets which forms a basis of $\Re^2$. No justification necessary.

(a) $\left\{\begin{bmatrix}1\\1\end{bmatrix}, \begin{bmatrix}1\\2\end{bmatrix}, \begin{bmatrix}3\\1\end{bmatrix}\right\}$, (b) $\left\{\begin{bmatrix}1\\1\end{bmatrix}, \begin{bmatrix}2\\2\end{bmatrix}\right\}$, (c) $\left\{\begin{bmatrix}1\\1\end{bmatrix}\right\}$, (d) $\underline{\left\{\begin{bmatrix}1\\1\end{bmatrix}, \begin{bmatrix}1\\2\end{bmatrix}\right\}}$, (e) $\left\{\begin{bmatrix}1\\1\end{bmatrix}, \begin{bmatrix}0\\0\end{bmatrix}\right\}$.

5. (a) (8 pts) Find all solutions to $\begin{bmatrix} 1 & 1 & 2 & 3 \\ 1 & 2 & 3 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$. Write your answer in vector form.

$$\left[\begin{array}{cccc|c} 1 & 1 & 2 & 3 & 0 \\ 1 & 2 & 3 & 0 & 0 \end{array}\right] \longrightarrow \left[\begin{array}{cccc|c} 1 & 1 & 2 & 3 & 0 \\ 0 & 1 & 1 & -3 & 0 \end{array}\right] \longrightarrow \left[\begin{array}{cccc|c} 1 & 0 & 1 & 6 & 0 \\ 0 & 1 & 1 & -3 & 0 \end{array}\right] \Rightarrow \begin{array}{l} x_3 = t \\ x_4 = s \end{array}$$

$x_1 + t + 6s = 0 \Rightarrow x_1 = -t - 6s$

$x_2 + t - 3s = 0 \Rightarrow x_2 = -t + 3s$

$$\therefore \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} -t-6s \\ -t+3s \\ t \\ s \end{bmatrix} = t\begin{bmatrix} -1 \\ -1 \\ 1 \\ 0 \end{bmatrix} + s\begin{bmatrix} -6 \\ 3 \\ 0 \\ 1 \end{bmatrix}$$

(b) (2 pts) What is the dimension of the set of solutions to part (a)?

2 (2 free vars)

6. (4 pts) Let $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$. Define a matrix B so that $BA = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ 3a_{21} & 3a_{22} & 3a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$

$$\underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 1 \end{bmatrix}}_{B} \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ 3a_{21} & 3a_{22} & 3a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

7. Consider the set of vectors $\left\{ \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 3 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \right\}$.

(a) (4 pts) Write down equation(s) that would need to be solved to determine whether or not this set of three vectors is linearly independent in $\Re^3$.

$$a\begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix} + b\begin{bmatrix} 1 \\ 1 \\ 3 \end{bmatrix} + c\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

(b) (4 pts) Write your equation(s) in the form of $A\vec{x} = \vec{b}$, where A is a matrix, and $\vec{x}$ and $\vec{b}$ are column vectors.

$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 2 & 3 & 0 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

(c) (4 pts) DO NOT SOLVE the system, but state what properties of the solution would determine whether the set of three vectors is linearly independent.

If this system has a ! s/n, then it is $a=0, b=0, c=0$ & the 3 vecs are indep

If " " " ∞ " , " the 3 vecs are dep.

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or/ If $\det(A) \neq 0 \Rightarrow$ independent

if $\det(A) = 0 \Rightarrow$ dependent

8. (6 pts) (True or False) $\begin{bmatrix}2\\2\\2\end{bmatrix}$ is in the span of $\left\{\begin{bmatrix}1\\1\\0\end{bmatrix}, \begin{bmatrix}0\\1\\1\end{bmatrix}\right\}$. Justify using the definition of span.

$$a\begin{bmatrix}1\\1\\0\end{bmatrix}+b\begin{bmatrix}0\\1\\1\end{bmatrix}=\begin{bmatrix}2\\2\\2\end{bmatrix}$$

Top eqn: $a\cdot 1 + b\cdot 0 = 2 \Rightarrow a = 2$

$\Rightarrow \begin{bmatrix}2\\2\\0\end{bmatrix} + b\begin{bmatrix}0\\1\\1\end{bmatrix} = \begin{bmatrix}2\\2\\2\end{bmatrix}$ last eqn. $\Rightarrow$ no sln.

$\therefore \begin{bmatrix}2\\2\\2\end{bmatrix}$ is not in the span of $\left\{\begin{bmatrix}1\\1\\0\end{bmatrix}, \begin{bmatrix}0\\1\\1\end{bmatrix}\right\}$

9. (10 pts) Consider the following subset W of $\Re^2$. PROVE that W is a vector subspace of $\Re^2$.

$$W = \left\{\begin{bmatrix}x_1\\x_2\end{bmatrix} \in \Re^2 : x_1 = x_2\right\}$$

1. Let $\vec{w} \in W. \Rightarrow \vec{w} = \begin{bmatrix}a\\a\end{bmatrix}$. Let $c \in \mathbb{R}$

$\Rightarrow c\vec{w} = c\begin{bmatrix}a\\a\end{bmatrix} = \begin{bmatrix}ca\\ca\end{bmatrix} \in W$ since its two coord's match

2. Let $\vec{u}, \vec{v} \in W \Rightarrow \vec{u} = \begin{bmatrix}a\\a\end{bmatrix}, \vec{v} = \begin{bmatrix}b\\b\end{bmatrix} \Rightarrow \vec{u} + \vec{v} = \begin{bmatrix}a+b\\a+b\end{bmatrix} \in W.$

$1+2 \Rightarrow$ W is a subspace of $\mathbb{R}^2$.

10. Extra Credit (+6 pts) Let $W = span\{1, x, 1+x\}$. It turns out that W is a (generalized) vector subspace of the set of all functions. Give a basis for W. Explain briefly, but a formal proof is not required.

$\mathcal{B} = \{1, x\}$ $\qquad$ $1+x$ is a lin. comb. of $\{1, x\}$ so it is not needed.