Math 3280 Practice Test 3. Directions: SHOW ALL WORK. Please do not write on the provided Laplace Transform tables. Indicate clearly any places where you use either the Laplace transform tables or a calculator.

1. (6 pts) From Calculus II, you can determine the solution to \( y'' = 1 \) by integration. Do this to find the general solution. Identify in this solution \( y_c \) (the solution to the corresponding homogeneous differential equation) and \( y_p \), the particular solution.

2. (6 pts) Find one particular solution to \( y'' + 2y' + 4y = 3e^{2x} \). (You need not find the general solution.)

3. (6 pts) Find the general solution to the constant coefficient linear homogeneous differential equation which has the following characteristic polynomial: \((r^2 + 4)(r - 4)(r + 4)^2\) (obtained by trying a solution of the form \( y(x) = e^{rx}\)).

4. (6 pts) What differential operator annihilates \( 5 \cos(3x) \)? Verify your answer.

5. (6 pts) Find the form of a particular solution to \( y'' - y' - 2y = 3e^{2t} \). Do not include extraneous terms and do not evaluate the “undetermined coefficients.”

6. (6 pts) What is the Laplace transform of \( g(t) = 3te^{2t} - u(t - 4) \)? (You may use the tables. You need not simplify your answer.)

7. (6 pts) Use the definition of the Laplace transform (not the tables) to show that the Laplace transform of \( e^{3t} \) is \( \frac{1}{s - 3} \).

8. (6 pts) Define \( f(t) = \begin{cases} 3 & 0 \leq t < 3 \\ t^2 & 3 \leq t < 4 \\ \cos(t) & 4 \leq t. \end{cases} \) Use step functions to write \( f(t) \) as a single line formula.

9. (6 pts) Solve using the method of Laplace transforms: \( y'(t) - 4y(t) = 0, \ y(0) = 2 \).

10. (7 pts) Compute the Laplace transform of the solution of the initial value problem:

\[
y'' + y' + 2y = 3\cos(4t); \quad y(0) = 0, \ y'(0) = 3.
\]

(Find only \( Y(s) \), not \( y(t) \).) Write your answer as a polynomial (in \( s \)) over a polynomial. You need not simplify your answer.
11. (7 pts) Find the inverse Laplace transform of \( F(s) = \frac{3s - 5}{s^2 + s - 12} \).

12. (7 pts) Write the differential equation \( y'' - 6y' - 2y = \cos(2t) \) with initial conditions \( y(0) = 3, y'(0) = 4 \) as an equivalent system of first order differential equations. Write the system in vector form: \( \dot{x} = Ax + b \). Write the initial conditions in vector form as well.

13. Let \( A = \begin{bmatrix} 1 & 2 \\ 2 & -2 \end{bmatrix} \).

   (a) (4 pts) The matrix \( A \) has an eigenvalue 2 with corresponding eigenvector \( \begin{bmatrix} 2 \\ 1 \end{bmatrix} \). Verify this using the definition of an eigenvalue and eigenvector.

   (b) (4 pts) Use part (a) along with the fact that \( A \) has a second eigenvalue of \(-3\) with corresponding eigenvector \( \begin{bmatrix} 1 \\ -2 \end{bmatrix} \) to find the general solution to \( \dot{x} = \begin{bmatrix} 1 & 2 \\ 2 & -2 \end{bmatrix} x \).

   (c) (4 pts) Use both parts (a) and (b) to determine the solution to \( \dot{x} = \begin{bmatrix} 1 & 2 \\ 2 & -2 \end{bmatrix} x \) which also satisfies the initial conditions \( x(0) = \begin{bmatrix} 5 \\ 0 \end{bmatrix} \).

14. (7 pts) Let \( A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \). Find any one eigenvalue and corresponding eigenvector for \( A \).

15. Consider the system of differential equations

\[
\dot{x} = 2x - x^2 - xy, \quad \dot{y} = -y + xy
\]

   (a) (3 pts) Find any one equilibrium point.

   (b) (3 pts) Compute and sketch in the \((x, y)\) phase plane the two velocity vectors at \((x, y) = (2, 1)\) and \((x, y) = (1, 1/2)\).

16. (Extra credit 6 pts) Use the definition of the Laplace transform to show that if \( F(s) \) is the Laplace transform of \( f(t) \), then the transform of \( f'(t) \) is \( sF(s) - f(0) \). (You need not provide details why “plugging in at \( \infty \)” is zero.)