Differential Equations with Linear Algebra, Math 3280
Labs #4,5,6
Applications of First Order Differential Equations: Numerical, Graphical, and Formula solutions. Interpretation of Mathematical Models.

B. Peckham

Directions: The following differential equations modelling problems are intended to supplement the modeling problems in chapters 1 and 2 in Edwards and Penny. Turn in a written lab report making sure to include in your writeup the items in the "Lab Procedures and Guidelines" handout. Make sure it is clear what steps you have done by computer and what steps you have done by hand.

Below are listed three modeling problems. For each one:

1. Determine an appropriate differential equation(s) to model the problem. Make sure you label all variables and provide an explanation of how you obtained your model. Also determine appropriate initial condition(s). It may be useful to "rescale" your variables (for example, measuring in thousands rather than ones).

2. Do the following **independently** for each of the four Methods listed below.

   (a) Solve the IVP(s) you came up with in the previous step.

   (b) Interpret your solution(s) in the context of the application. Discuss whether the solutions seem consistent with your intuition about the application. In particular, discuss long term predictions and the meaning of any equilibria (if appropriate). Also discuss the time interval for which your solutions are valid.

   (c) Use your solution(s) to answer any additional questions asked about the solution.

Compare results obtained by the four methods. Discuss the effort involved and the accuracy of solutions from each method. Explain any differences.

The Solution Methods:

- Method 1: If appropriate, sketch a phase line for the problem. If not appropriate, state why. If phase lines have been drawn, use them to obtain a *qualitative* sketch of several representative solutions. Include the solution corresponding to the initial condition(s) for the model.

- Method 2: Use slope fields to obtain a graphical solution corresponding to the appropriate initial condition(s). You may obtain the slope field(s) either by hand or using *Mathematica*.

- Method 3: Solve the problem numerically, using the NDSolve command in *Mathematica*. Include at least the solution corresponding to the initial condition(s) for the model. Include a plot of your numerically obtained solution(s).

- Method 4: Solve each problem analytically (obtain a formula solution). You are encouraged to have Mathematica do all integrations and algebraic manipulations (solving for variables, simplifying) necessary to obtain the analytic solution. Using separation of variables or linear first order templates is acceptable. Include a plot of your analytic solution for comparison with the other methods.

The Problems:

1. **Falling object.** Lab 4. An object(possibly a person) is released from a helocopter which is at rest (hovering 1000 meters above the ground). The object falls toward the earth. Assume that the only forces acting on the object are gravity (with a force of mg) and air resistance, which is assumed to be proportional to the cube of the velocity, with proportionality constant 0.05. Assume the mass m of the object is $m = 50$ kg, and the acceleration due to gravity is constant during the fall at $g = -9.81 m/s^2$. (NOTE: The choice of g as -9.81 instead of $+9.81$ means I must have chosen UP as the positive direction. This is important for interpretation of the solution. Think also what it means for the sign of the air resistance force.) Does the differential equation change if the object is going up versus down? What is the velocity as a function of time? What is the "long-term" behavior of the velocity?

   NOTE: Mathematica is able to do the integrations necessary to solve this problem using separation of variables, but (at least for the instructor) Mathematica's Solve command failed to convert the implicit solution to an explicit solution. However, you can solve for t in terms of v. You will be able to plot this in the v-t plane as opposed to the solutions for methods 1, 2 and 3 where solutions were plotted with v as a function of t. A plot of t(v) is acceptable for displaying the analytic solution. This difficulty should appear in your comparison of the four methods for this problem.

   Extra Credit:

   - Find the time at which the object hits the ground. Since you cannot obtain the velocity as a function of time, then you cannot "just integrate" to get the position function as a function of time. Consequently, you must be creative. Any logical estimate is better than no answer! Explain your answer.
   - For any of the 4 methods, find, sketch, or plot the position as as function of t.

2. **Compartmental analysis** Lab 5.A tank whose volume is 200 L initially contains 100 L of water and 100 grams of salt. A solution containing 0.5 g/L of salt is pumped into the tank at a rate of 6 L/min, and the well-stirred mixture flows out at a rate of 4 L/min. How much salt is in the tank just before the solution overflows. How much salt is in the tank 10 minutes after it starts overflowing? (Careful!! Answering this question requires using a second differential equation!! How can you show your solutions on a single graph? Some "cutting and pasting" may be useful.)

3. **Bacteria growth and "parameter matching."** Lab 6.Assume that the number of bacteria in a culture grows at a rate that is proportional to the amount present. After 5 hours there were 50 grams of bacteria present, and after 10 hours there were 60 grams of bacteria present. Determine the initial size of the culture and the doubling time of the population. For numerical solutions (method 3), you can determine the constant of proportionality either by trial and error, or by using the Manipulate command.

Grading for each problem: Goals (G) 2pts, Procedures (Pro) 2pts, model and justification 3pts, four solution methods 10pts, Compare methods 4pts, Questions/Conclusions (C) 5pts, Presentation and Organization (P+O) 4pts, Total 30.