Math 3280
Differential Equations with Linear Algebra
Answers to Sample 'Operator' and 'Modeling' Test Questions

1. Define the operator $L : C^1(-\infty,\infty) \to C^0(-\infty,\infty)$ by $L = D - 2x$, where $D = \frac{d}{dx}$. Compute $L[x^2]$ and $L^2[x^2]$.

   $2x - 2x^3$, $2 - 10x^2 + 4x^4$

2. Assume that L is known to be a linear operator. Assume that it is known that $L[f(t)] = 3e^{2t}$ and $L[g(t)] = 6e^{2t}$. Write down a solution to $L[y] = 0$ in terms of $f(t)$ and $g(t)$. Justify your answer.

   $g(t) - 2f(t)$ is one of many possible answers.

3. What is the general solution to

   (a) $(D^2 - 1)[y] = 0$.

   $c_1 e^t + c_2 e^{-t}$

   (b) $((D-3)^2 + 4)^2 (D-1)D[y] = 0$.

   $c_1 e^t + c_2 e^{3t}\cos(2t) + c_3 e^{3t}\sin(2t) + c_4 t e^{3t}\cos(2t) + c_5 t e^{3t}\sin(2t) + c_6$

4. What is an annihilator of

   (a) e^{-2x}

   $D + 2$

   (b) $x^2 e^{5x} + \cos(2x) + 1$?

   $(D-5)^3(D^2+4)D$

5. (8 pts) Use the method of annihilators to find the form of a trial solution for the particular solution $y_p(x)$ for $(D^2+1)(D-3)[y] = 3\sin(x)$. Do not include extraneous terms. Show your work.

   $y_p(x) = Ax\sin(x) + Bx\cos(x)$

6. Define the operator $L = D^2 + 4$, where $D = \frac{d}{dx}$.

   (a) What functions of the form $A + Bx$ are solutions to the differential equation $L[y] = 5x$?

   $y(x) = 0 + \frac{5}{4}x$

   (b) If $L[y_1] = 2\sin(x)$ and $L[y_2] = 5e^{5x}$, what is $L[3y_1 - y_2]$?

   $6\sin(x) - 5e^{5x}$

   (c) If $\phi(x)$ is a solution to $L[y] = 5e^{5x}$, is $2\phi(x)$ also a solution? Explain.

   No.

7. What constant coefficient linear homogeneous differential equation does the function

$$xe^{3x}\cos(2x) + 4$$

satisfy? (Hint: Think of annihilators.)

$((D-3)^2+4)^2 D[y] = 0$

8. (2 pts) Assume the temperature $T(t)$ inside a building behaves according to Newton's law: $\dot{T}(t) = k(T(t) - T_o)$, where T_o is the outside temperature. Is the constant k positive or negative? Why?

Negative.

9. (8pts) Saltwater with a concentration of .01kg/L flows at a rate of 3 L/min into a 400L tank that initially held 200 L of pure water. The tank is kept well stirred and flows out of the tank at a rate of 2 L/min. Let $x(t)$ denote the amount in kilograms of salt in the tank at time t. Carefully write a differential equation, with initial condition, that describes the change in amount of salt in the tank as a function of time. Label your variables. (Do **not** solve.) Over what time interval would the differential equation be valid? Why?

$\dot{x} = (.01)(3) - 2\frac{x}{200+t}$, $x(0) = 0$. Valid on $0 \le t \le 200$.

10. (7pts) Write down a differential equation which models the motion of a parachutist. Make the following assumptions:

(a) $F = ma$.

(b) The force due to gravity is constant.

(c) The parachute opens immediately as the parachutist steps out of the plane.

(d) The force exerted by the open parachute is proportional to the **cube** of the speed of the falling parachutist.

(e) There are no external forces (other than gravity) acting.

Write a differential equation **with initial conditions** which models the parachutist's fall. Label the variables you use. Write your differential equation so that any constants you use are positive.

$m\dot{v} = mg - bv^3$, $v(0) = 0$.