Math 3280 Differential Equations with Linear Algebra
Lab #7: Resonant Forcing of a Spring-mass system

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Tues. October 31 and Nov. 7, 2006
Due: Wed. Nov. 8, 2006

Directions: Turn in a written lab report dealing with the tasks below. You may decide what tasks to do with Mathematica and which to do by hand. Be sure to include in your writeup the items in the “Lab Procedures and Guidelines” handout. Previous Lab handouts may be useful. Material in Section 5.1, 5.3, 5.4 and especially 5.6 of Edwards and Penny may also be of help.

The motion of a mass on a spring with an external forcing function is determined by the model differential equation:

$$25\frac{d^2x}{dt^2} + 10\frac{dx}{dt} + 50x = F_0\cos(\gamma t) \tag{1}$$

The general solution to (1) turns out to be:

$$x(t) = c_1 e^{\frac{-1}{5}t}\cos(\frac{7}{5}t) + c_2 e^{\frac{-1}{5}t}\sin(\frac{7}{5}t) + \frac{F_0}{100-96\gamma^2+25\gamma^4}[(2-\gamma^2)\cos(\gamma t) + \frac{2}{5}\gamma\sin(\gamma t)] \tag{2}$$

1. Check that the solution given in equation (2) is actually the general solution. (That is, identify “x_1, x_2, and x_p” and check that they are solutions to the appropriate differential equations; also check to see that the Wronskian determinant of x_1 and x_2 is not “identically zero.” You should know how to obtain the general solution even though you are not required to ”solve” from scratch for this lab.)

2. Resonant behavior:

   (a) Why is the particular part of the solution the most important part of the solution (especially in the long run)?

   (b) Show that the particular solution in equation (2) above can be rewritten in the form $x_p(t) = F_0 M(\gamma)\sin(\gamma t + \theta)$, where

   $$M(\gamma) = \frac{\sqrt{(2-\gamma^2)^2 + (\frac{2\gamma}{5})^2}}{100-96\gamma^2+25\gamma^4}$$

   (Hint: If $A\cos(\gamma t) + B\sin(\gamma t) = C\sin(\gamma t + \theta)$, then use the trig. identity $\sin(\gamma t+\theta) = \sin(\gamma t)\cos(\theta) + \cos(\gamma t)\sin(\theta)$ to show that $C = \sqrt{A^2+B^2}$. You do not have to explicitly solve for θ.)

   (c) The “new” form of x_p makes it obvious that x_p is an oscillation with amplitude $F_0 M(\gamma)$. In particular, it shows that the “output amplitude” depends on the “input frequency.” This dependence, $M(\gamma)$, is called the “frequency response curve.” Plot it for $0 \le \gamma \le 5$. Determine from this plot the approximate value (eyeball it) of γ_r which makes $M(\gamma)$ a maximum. This value of γ_r is called the *resonant frequency* of the spring-mass system. Why is it important?

   (d) Assuming $F_0 = 4$, use the graph of $M(\gamma)$ to predict the approximate output amplitude of the particular solution to equation (1) for the three values of γ: $0.5, 1.386$, and 3.0.

   (e) Obtain and graph the three particular solutions (of the form used in part (b) or in equation (2), with $c_1 = 0, c_2 = 0$) for $F_0 = 4$, and the three γ values of $0.5, 1.386$, and 3.0. Graph the three solutions on the same set of axes and compare them. Compare them also to your predictions of the “response amplitudes” in part (d).