1. Consider the following problems in the vector space $\mathbb{R}^3$. Let $\vec{a} = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}$, and $\vec{b} = \begin{bmatrix} 0 \\ 2 \\ 1 \end{bmatrix}$. Let $L = \text{span}\{\vec{b}\}$. Let $W = \text{span}\{\vec{a}, \vec{b}\}$.

(a) (3pts) Compute $\vec{a} \bullet \vec{b}$ $= 1\cdot 0 + 2\cdot 2 + 0\cdot 1 = 4$

(b) (3pts) Compute $||\vec{a}||$ $= \sqrt{1^2 + 2^2 + 0^2} = \sqrt{5}$

(c) (4pts) Compute $\text{proj}_L \vec{a}$ $= \dfrac{\vec{a}\cdot\vec{b}}{\|\vec{b}\|^2}\vec{b} = \dfrac{4}{5}\begin{bmatrix} 0 \\ 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 8/5 \\ 4/5 \end{bmatrix}$

(d) (4pts) Compute $\text{proj}_{L^\perp}\vec{a}$ (the projection of $\vec{a}$ onto $L^\perp$)

$= \vec{a} - \text{proj}_L \vec{a} = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} - \begin{bmatrix} 0 \\ 8/5 \\ 4/5 \end{bmatrix} = \begin{bmatrix} 1 \\ 2/5 \\ -4/5 \end{bmatrix}$

(e) (4pts) Compute $\text{proj}_W \vec{a}$ $= \vec{a}$ since $\vec{a} \in W$.

(f) (6pts) Compute an orthonormal basis for W.

Use Gram-Schmidt to get orthogonal basis: Start with either $\vec{a}$ or $\vec{b}$. (I chose $\vec{a}$.)

Let $\vec{b}_1 = \vec{a} = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}$. Let $\vec{b}_2 = \vec{b} - \text{proj}_{\vec{b}_1} \vec{b} = \begin{bmatrix} 0 \\ 2 \\ 1 \end{bmatrix} - \dfrac{\vec{b}\cdot\vec{a}}{\|\vec{a}\|^2}\begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}$

$= \begin{bmatrix} 0 \\ 2 \\ 1 \end{bmatrix} - \dfrac{4}{5}\begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} = \begin{bmatrix} -4/5 \\ +2/5 \\ 1 \end{bmatrix}$. Normalize to get $\vec{u}_1 = \dfrac{\vec{b}_1}{\|\vec{b}_1\|}$, $\vec{u}_2 = \dfrac{\vec{b}_2}{\|\vec{b}_2\|}$:

ie, $\boxed{\vec{u}_1 = \dfrac{1}{\sqrt{5}}\begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}, \quad \vec{u}_2 = \dfrac{1}{\sqrt{45}}\begin{bmatrix} -4 \\ 2 \\ 5 \end{bmatrix}.}$

2. Consider the system of equations:

$$
\begin{aligned}
x &= 1 \\
y &= 1 \\
x + y &= 1
\end{aligned}
$$

(a) (3pts) Write the system in the form $A\vec{x} = \vec{b}$.

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

(b) (3pts) Explain briefly why the system is inconsistent.

If the 1st 2 eqns are satisfied, the third isn't.

or $\left[\begin{array}{cc:c} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 1 \end{array}\right] \to \left[\begin{array}{cc:c} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 1 & 0 \end{array}\right] \to \left[\begin{array}{cc:c} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & -1 \end{array}\right]$ last row $\Rightarrow$ no sln.

(c) (10pts) Find a least squares solution ($\hat{x}$) to $A\vec{x} = \vec{b}$ and the least squares error for this solution (the distance from $A\hat{x}$ to $\vec{b}$). You may answer these two questions in either order. Label your work clearly.

Method 1: $\hat{b} = \operatorname{proj}_{\operatorname{Col}A} \vec{b}$. 1st need orthog basis for Col A.

$\vec{x}_1 = \vec{u}_1 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$. $\vec{u}_2 = \vec{x}_2 - \frac{\vec{x}_2 \cdot \vec{u}_1}{\|\vec{u}_1\|^2} \vec{u}_1 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} - \frac{1}{2} \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} -\frac{1}{2} \\ 1 \\ \frac{1}{2} \end{bmatrix}$

$\Rightarrow \hat{b} = \frac{\vec{b} \cdot \vec{u}_1}{\|\vec{u}_1\|^2} \vec{u}_1 + \frac{\vec{b} \cdot \vec{u}_2}{\|u_2\|^2} \vec{u}_2 = \frac{2}{2} \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} + \frac{1}{(\frac{1}{2})^2 + 1 + (\frac{1}{2})^2} \begin{bmatrix} -\frac{1}{2} \\ 1 \\ \frac{1}{2} \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} + \frac{2}{3} \begin{bmatrix} -\frac{1}{2} \\ 1 \\ \frac{1}{2} \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} + \begin{bmatrix} -\frac{1}{3} \\ \frac{2}{3} \\ \frac{1}{3} \end{bmatrix}$

$= \begin{bmatrix} 2/3 \\ 2/3 \\ 4/3 \end{bmatrix} = \hat{b}$

Solve: $A \begin{bmatrix} x \\ y \end{bmatrix} = \hat{b}$: $\left[\begin{array}{cc:c} 1 & 0 & 2/3 \\ 0 & 1 & 2/3 \\ 1 & 1 & 4/3 \end{array}\right] \to \begin{bmatrix} 1 & 0 & \frac{2}{3} \\ 0 & 1 & 2/3 \\ 0 & 0 & 0 \end{bmatrix} \Rightarrow \hat{x} = \begin{bmatrix} 2/3 \\ 2/3 \end{bmatrix}$

$\|\vec{b} - \hat{b}\| = \sqrt{(\frac{2}{3})^2 + (\frac{2}{3})^2 + (\frac{2}{3})^2}$
$= 2\sqrt{3}/2$

Method 2: Solve $A^T A \vec{x} = A^T \vec{b}$: $\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ for $\hat{x}$.

i.e., $\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \end{bmatrix}$. By row red: $\left[\begin{array}{cc:c} 2 & 1 & 2 \\ 1 & 2 & 2 \end{array}\right] \to \begin{bmatrix} 1 & 2 & 2 \\ 0 & -3 & -2 \end{bmatrix} \to \begin{bmatrix} 1 & 2 & 2 \\ 0 & 1 & \frac{2}{3} \end{bmatrix} \Rightarrow y = \frac{2}{3}$, $x + 2 \cdot \frac{2}{3} = 2$
$\Rightarrow x = \frac{2}{3}$

$\therefore \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 2/3 \\ 2/3 \end{bmatrix} = \hat{x}$ and $\hat{b} = A\hat{x} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 2/3 \\ 2/3 \end{bmatrix} = \begin{bmatrix} 2/3 \\ 2/3 \\ 4/3 \end{bmatrix}$.

L.Sq. error $= \left\| \begin{bmatrix} 2/3 \\ 2/3 \\ 4/3 \end{bmatrix} - \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \right\| = \sqrt{\frac{1}{3}^2 + \frac{1}{3}^2 + \frac{1}{3}^2} = 1/\sqrt{3} = \frac{\sqrt{3}}{3}$

5. (8pts) Orthogonally diagonalize the matrix $A = \begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix}$. (That is, find a matrix P such that $P^{-1}AP$ is diagonal.)

Evals: $\begin{vmatrix} 3-\lambda & 1 \\ 1 & 3-\lambda \end{vmatrix} = \lambda^2 - 6\lambda + \cancel{4}9 - 1 = \lambda^2 - 6\lambda + 8 = (\lambda-2)(\lambda-4) \Rightarrow \lambda = 2, 4$

For $\lambda = 2$: $\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \vec{0} \Rightarrow v_1 = -v_2 \Rightarrow \vec{v} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$. or unit vector $\begin{bmatrix} 1/\sqrt{2} \\ -1/\sqrt{2} \end{bmatrix}$

For $\lambda = 4$ $\begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix}\vec{v} = \vec{0} \Rightarrow v_1 = v_2 \Rightarrow \vec{v} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ or unit vector $\begin{bmatrix} 1/\sqrt{2} \\ +1/\sqrt{2} \end{bmatrix}$. So $P = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ -1 & +1 \end{bmatrix}$

6. Consider the vector space $\mathbb{P}_4$ (polynomials of degree 4 or less) with the inner product $\langle p_1, p_2 \rangle = \int_{-1}^{1} p_1(t)p_2(t)dt$. Let $f(t) = t^2$, and $g(t) = t^4$.

(a) (4pts) Compute $\langle f, g \rangle$ $= \int_{-1}^{1} t^2 \cdot t^4 \, dt = \int_{-1}^{1} t^6 \, dt = \frac{t^7}{7}\Big|_{-1}^{1} = \frac{1}{7} - \frac{-1}{7} = \frac{2}{7}$

(b) (6pts) What is the angle between f and g? (You need not simplify your answer; for example it can be the *arccos* of some expression.)

$\langle f, g \rangle = \|f\| \, \|g\| \cos\theta$

$\|f\| = \sqrt{\int_{-1}^{1} t^4 \, dt} = \sqrt{\frac{2}{5}}$, $\|g\| = \sqrt{\int_{-1}^{1} t^4 \cdot t^4 \, dt} = \sqrt{\frac{2}{9}}$

$\therefore \frac{2}{7} = \sqrt{\frac{2}{5}}\sqrt{\frac{2}{9}} \cos\theta = \frac{2}{3\sqrt{5}} \cos\theta \Rightarrow \cos\theta = \frac{\cancel{2}}{7} \cdot \frac{3\sqrt{5}}{\cancel{2}} = \frac{3\sqrt{5}}{7}$

$\therefore \theta = \arccos 3\sqrt{5}/7$

7. (8pts) Let A be an $m \times n$ matrix. Show that if $\vec{x} \in Col(A)^{\perp}$, then $\vec{x} \in Nul(A^T)$. Hint: choose notation for the columns of A.

Let $\vec{x} \in (Col\, A)^{\perp}$. Let $A = [\vec{a}_1 \cdots \vec{a}_n]$, so $\vec{a}_i$ is the i^{th} column of A.

$\vec{x} \in (Col\, A)^{\perp} \Rightarrow \vec{x} \cdot \vec{a}_i = 0$ for $i = 1, \ldots, n$.

$\Rightarrow A^T \cdot \vec{x} = \begin{bmatrix} \vec{a}_1^T \\ \vdots \\ \vec{a}_n^T \end{bmatrix} \vec{x} = \begin{bmatrix} a_1^T \vec{x} \\ \vdots \\ a_n^T \vec{x} \end{bmatrix} = \begin{bmatrix} \vec{a}_1 \cdot \vec{x} \\ \vdots \\ \vec{a}_n \cdot \vec{x} \end{bmatrix} = \begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix} = \vec{0} \Rightarrow \vec{x} \in Nul(A^T)$

3. (8pts) Let $A = \begin{bmatrix} 1 & -2 \\ 1 & 3 \end{bmatrix}$. Find an invertible matrix P for which $P^{-1}AP$ is of the form: $\begin{bmatrix} a & -b \\ b & a \end{bmatrix}$. Identify a and b.

Evals of A: $\begin{vmatrix} \lambda - 1 & 2 \\ -1 & \lambda - 3 \end{vmatrix} = \lambda^2 - 4\lambda + 3 + 2 = \lambda^2 - 4\lambda + 5$

$\Rightarrow \lambda = \frac{4 \pm \sqrt{16-20}}{2} = 2 \pm i$

$\lambda = 2+i$: $\begin{bmatrix} (2+i)-1 & 2 \\ -1 & (2+i)-3 \end{bmatrix} \vec{v} = \begin{bmatrix} 1+i & 2 \\ -1 & -1+i \end{bmatrix} \vec{v}$

$\Rightarrow \vec{v} = \begin{bmatrix} -2 \\ 1+i \end{bmatrix} = \begin{bmatrix} -2 \\ 1 \end{bmatrix} + i \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ or $\begin{bmatrix} -2+2i \\ 2 \end{bmatrix}$ or $\begin{bmatrix} -1+i \\ 1 \end{bmatrix}$ or $\begin{bmatrix} 1-i \\ -1 \end{bmatrix}$

$\Rightarrow P = \begin{bmatrix} -2 & 0 \\ 1 & 1 \end{bmatrix}$. $\Rightarrow P^{-1}AP = \begin{bmatrix} 2 & 1 \\ -1 & 2 \end{bmatrix} \Rightarrow a = 2, b = -1.$

4. $A = \begin{bmatrix} 2 & 3 \\ 0 & 2 \end{bmatrix}$.

(a) (7pts) Find the $\overset{2\text{pts}}{\text{eigenvalues}}$ and $\overset{5\text{pts}}{\text{singular values}}$ for A.

By inspection, $\lambda = 2, 2$.

$A^TA = \begin{bmatrix} 2 & 0 \\ 3 & 2 \end{bmatrix}\begin{bmatrix} 2 & 3 \\ 0 & 2 \end{bmatrix} = \begin{bmatrix} 4 & 6 \\ 6 & 13 \end{bmatrix}$

Evals of A^TA: $\begin{vmatrix} 4-\lambda & 6 \\ 6 & 13-\lambda \end{vmatrix} = \lambda^2 - 17\lambda + 52 - 36$
$= \lambda^2 - 17\lambda + 16$
$= (\lambda - 16)(\lambda - 1)$
$\Rightarrow \lambda = 16, 1$

$\Rightarrow$ Singular values are $\sqrt{16} = 4$ and $\sqrt{1} = 1$.

(b) (3pts) Find the maximum value of $||A\vec{x}||$ restricted to $\vec{x}$ satisfying $||\vec{x}|| = 1$. Explain briefly.

Max expansion is max singular value: 4.

Illustration of max expansion:

Not needed: evec of A^TA for $\lambda = 16$: $\begin{bmatrix} -12 & 6 \\ 6 & -3 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow \vec{v} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$

$A\begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 2 & 3 \\ 0 & 2 \end{bmatrix}\begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 8 \\ 4 \end{bmatrix}$

$\frac{\left\|\begin{bmatrix} 8 \\ 4 \end{bmatrix}\right\|}{\left\|\begin{bmatrix} 1 \\ 2 \end{bmatrix}\right\|} = \frac{\sqrt{64+16}}{\sqrt{1+4}} = \frac{4\sqrt{5}}{\sqrt{5}} = 4$

8. (8pts) Let W be a subspace of the vector space $\mathbb{R}^n$. Let $W^\perp = \{\vec{x} \in \mathbb{R}^n : \vec{x} \bullet \vec{w} = 0 \; \forall w \in W\}$ Prove that $W^\perp$ is a vector subspace of $\mathbb{R}^n$.

Let $\vec{a} \in W^\perp, \vec{b} \in W^\perp, c \in \mathbb{R}, \vec{w} \in W$.

i $\vec{0} \cdot \vec{w} = 0 \quad \forall \vec{w} \in W \Rightarrow \vec{0} \in W^\perp$

ii $(\vec{a}+\vec{b}) \cdot \vec{w} = \vec{a} \cdot \vec{w} + \vec{b} \cdot \vec{w} = 0 + 0$ since $\vec{a}, \vec{b} \in W^\perp$.
$= 0 \Rightarrow \vec{a}+\vec{b} \in W^\perp$

iii $(c\vec{a}) \cdot \vec{w} = c(\vec{a} \cdot \vec{w}) = c \cdot 0$ since $\vec{a} \in W^\perp$
$= 0 \Rightarrow c\vec{a} \in W^\perp$

i, ii, iii $\Rightarrow$ $W^\perp$ is a subspace of $\mathbb{R}^n$.

9. (8pts) Let A be any $m \times n$ matrix. Prove that the eigenvalues of $A^T A$ are all real and nonnegative.

Let λ be an e-val of $A^T A$ w/ corresponding e-vec $\vec{v}$.

ie, $A^T A \vec{v} = \lambda \vec{v} \Rightarrow$

$\|A\vec{v}\|^2 = A\vec{v} \cdot A\vec{v} = (A\vec{v})^T(A\vec{v}) = \vec{v}^T(A^T A \vec{v}) = \vec{v}^T \lambda \vec{v} = \lambda \vec{v}^T \vec{v}$

$= \lambda \|\vec{v}\|^2 \Rightarrow \lambda = \frac{\|A\vec{v}\|^2}{\|\vec{v}\|^2} \geq 0$ and real.

ie, λ is real + non negative

10. (Extra Credit 5 pts) Consider the system $A\vec{x} = \vec{b}$ which is assumed to be inconsistent. Derive the "normal equations" $A^T A \vec{x} = A^T \vec{b}$ for $\hat{x}$ assuming $A\hat{x} = \hat{b}$ where $\hat{b}$ is the orthogonal projection of $\vec{b}$ onto the $Col(A)$. You may assume $(Col\, A)^\perp = Nul(A^T)$.

Decompose $\vec{b} = \hat{b} + \vec{z}$ where $\hat{b} \in Col\, A$, $\vec{z} \in (Col\, A)^\perp = Nul(A^T)$. $\hat{b} \in Col\, A \Rightarrow \exists \hat{x}$ s.t. $A\hat{x} = \hat{b}$.

$z \in Nul(A^T) \Rightarrow A^T\vec{z} = \vec{0}$ ie, $A^T(\vec{b} - \hat{b}) = \vec{0}$ ie, $A^T(\vec{b} - A\hat{x}) = \vec{0}$ or $A^T\vec{b} = A^T A\hat{x}$,

$\Rightarrow \hat{x}$ satisfies $A^T A \hat{x} = A^T \vec{b}$ //