1. (5 pts) Solve the following linear system USING GAUSSIAN ELIMINATION (row reduction to echelon or reduced echelon form). Leave your answers as exact fractions.

$$\begin{bmatrix} 1 & 3 \\ 1 & -3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$$

$$\left[\begin{array}{cc|c} 1 & 3 & 1 \\ 1 & -3 & 3 \end{array}\right] \longrightarrow \left[\begin{array}{cc|c} 1 & 3 & 1 \\ 0 & -6 & 2 \end{array}\right] \longrightarrow \left[\begin{array}{cc|c} 1 & 3 & 1 \\ 0 & 1 & -\frac{1}{3} \end{array}\right] \longrightarrow \left[\begin{array}{cc|c} 1 & 0 & 2 \\ 0 & 1 & -\frac{1}{3} \end{array}\right] \Rightarrow \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 2 \\ -\frac{1}{3} \end{bmatrix}$$

2. Let $A = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 4 \end{bmatrix}$.

(a) (5 pts) Compute A^{-1} using row reduction.

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 4 & 0 & 0 & 1 \end{array}\right] \longrightarrow \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & -1 & 1 & 0 \\ 0 & 0 & 4 & 0 & 0 & 1 \end{array}\right] \longrightarrow \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & -1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & \frac{1}{4} \end{array}\right] \Rightarrow A^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & \frac{1}{4} \end{bmatrix}$$

(b) (5 pts) Use your work in part (a) to write A^{-1} as the product of elementary matrices.

$$E_1 = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad E_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \frac{1}{4} \end{bmatrix} \qquad A^{-1} = (E_2)(E_1).$$

(c) (3 pts) Write A as a product of elementary matrices.

$$\Rightarrow A = E_1^{-1} E_2^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 4 \end{pmatrix}$$

3. (5 pts) Evaluate the following determinant. Show your work.

$$\begin{vmatrix} 0 & 3 & 1 & 2 \\ 1 & 0 & 3 & 1 \\ 1 & 0 & 1 & 1 \\ 0 & 0 & 2 & 3 \end{vmatrix} = (-3) \begin{vmatrix} 1 & 3 & 1 \\ 1 & 1 & 1 \\ 0 & 2 & 3 \end{vmatrix} = (-3)\left[1 \begin{vmatrix} 1 & 1 \\ 2 & 3 \end{vmatrix} - 1 \begin{vmatrix} 3 & 1 \\ 2 & 3 \end{vmatrix}\right] = (-3)(1 - 7) = (-3)(-6) = 18$$

4. (3 pts) If A is a 3×3 matrix, and $\det(A) = 1$, what is $\det(2A)$? Explain briefly.

$\det A = 2^3 \cdot 1$ Since there are 3 factors in each term in $\det(2A)$, each multiplied by 2.

5. (5 pts) Circle any of the following sets which forms a basis of $\Re^2$. No justification necessary.

(a) $\left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 3 \\ 1 \end{bmatrix} \right\}$, (b) $\left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 2 \end{bmatrix} \right\}$, (c) $\left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right\}$, (d) $\left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \end{bmatrix} \right\}$, (e) $\left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \end{bmatrix} \right\}$.

6. (5 pts) Write any system of equations in 5 variables, $x_1, x_2, ..., x_5$ which has a solution set which is a 3-dimensional subspace of $\Re^5$. You can decide how many equations to use.

$$\begin{aligned} x_1 &= 0 \\ x_2 &= 0 \end{aligned} \Rightarrow \text{free}: x_3, x_4, x_5 \Rightarrow \text{basis} \left\{ \begin{bmatrix}0\\0\\1\\0\\0\end{bmatrix}, \begin{bmatrix}0\\0\\0\\1\\0\end{bmatrix}, \begin{bmatrix}0\\0\\0\\0\\1\end{bmatrix} \right\}$$

7. (15 pts) TRUE-FALSE. (3pts for correct answers, 2pts for justification) If TRUE, give a brief explanation other than quoting a theorem. A formal proof is not required. If FALSE, give a counter example.

(a) The set of all solutions to

$$\begin{bmatrix} 2 & 1 & 4 \\ 3 & 3 & 2 \\ 6 & 0 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 2 \\ 1 \end{bmatrix}$$

is a vector subspace of $\Re^3$.

No: $\begin{bmatrix}0\\0\\0\end{bmatrix}$ is not a sln.
(FALSE)

(b) The columns of any 4×5 matrix are linearly dependent.

(TRUE) 5 vectors in $\mathbb{R}^4$ are always lin dep.

(c) If A is a 3×4 matrix with a pivot in each row, then $A\vec{x} = \vec{b}$ will be consistent for all $\vec{b}$.

TRUE, pivot in each row ⇒ cannot have a row of all zeros.
⇒ in an augmented matrix, no row of $0 \cdots 0 : \square$.
⇒ not inconsistent.

8. Consider the set of vectors $\left\{ \begin{bmatrix}1\\0\\2\end{bmatrix}, \begin{bmatrix}1\\1\\3\end{bmatrix}, \begin{bmatrix}1\\1\\0\end{bmatrix} \right\}$.

(a) (4 pts) Write down equation(s) that would need to be solved to determine whether or not this set of three vectors is linearly independent in $\Re^3$. Write your equation(s) in the form of $A\vec{x} = \vec{b}$, where A is a matrix, and $\vec{x}$ and $\vec{b}$ are column vectors.

$$c_1\begin{bmatrix}1\\0\\2\end{bmatrix} + c_2\begin{bmatrix}1\\1\\3\end{bmatrix} + c_3\begin{bmatrix}1\\1\\0\end{bmatrix} = \begin{bmatrix}0\\0\\0\end{bmatrix} \Leftrightarrow \begin{bmatrix}1 & 1 & 1\\0 & 1 & 1\\2 & 3 & 0\end{bmatrix}\begin{bmatrix}c_1\\c_2\\c_3\end{bmatrix} = \begin{bmatrix}0\\0\\0\end{bmatrix}$$

(b) (3 pts) DO NOT SOLVE the system, but state what properties of the solution would determine whether the set of three vectors is linearly independent.

If there is no s'n other than $c_1 = c_2 = c_3 = 0$, then the set is lin. indep.

9. (5 pts) Is $\begin{bmatrix}2\\2\\2\end{bmatrix}$ in the span of $\left\{\begin{bmatrix}1\\1\\0\end{bmatrix}, \begin{bmatrix}0\\1\\1\end{bmatrix}\right\}$? Justify using the definition of span.

Solve $c_1\begin{bmatrix}1\\1\\0\end{bmatrix} + c_2\begin{bmatrix}0\\1\\1\end{bmatrix} = \begin{bmatrix}2\\2\\2\end{bmatrix}$.

Top eq. $\Rightarrow c_1 = 2$, $3^{rd} \Rightarrow c_2 = 2$

but then $2^{nd} \Rightarrow 2+2=2$ $\therefore$ inconsistent!

or// $\begin{bmatrix}1 & 0 & 2\\1 & 1 & 2\\0 & 1 & 2\end{bmatrix} \to \begin{bmatrix}1 & 0 & 2\\0 & 1 & 0\\0 & 1 & 2\end{bmatrix} \to \begin{bmatrix}1 & 0 & 2\\0 & 1 & 0\\0 & 0 & 2\end{bmatrix} \Rightarrow$ inconsistent

$\therefore$ No

10. (5 pts) Assume that the three vectors $\vec{u}, \vec{v}, \vec{w}$ are in $\Re^n$, and that $3\vec{u} - 5\vec{v} - \vec{w} = \vec{0}$. Let $A = [\vec{u}\vec{v}]$. By inspection (without doing row reduction), Find a solution to $A\vec{x} = \vec{w}$.

$[\vec{u}\ \vec{v}]\begin{bmatrix}3\\-5\end{bmatrix} = \vec{w}$ since $[\vec{u}\ \vec{v}]\begin{bmatrix}3\\-5\end{bmatrix} = 3\vec{u} - 5\vec{v}$ which $= \vec{w}$.

ie, $\vec{x} = \begin{bmatrix}3\\5\end{bmatrix}$

11. (5 pts) Give an example of a 3×3 matrix A for which $\begin{bmatrix}1\\1\\1\end{bmatrix}$ is a solution to $A\vec{x} = \vec{0}$.

$$\underbrace{\begin{bmatrix}1 & 1 & -2\\2 & 3 & -5\\1 & -1 & 0\end{bmatrix}}_{A}\begin{bmatrix}1\\1\\1\end{bmatrix} = \begin{bmatrix}0\\0\\0\end{bmatrix}$$

12. Let $A = \begin{bmatrix}1 & 2 & 3 & 4\\1 & 2 & 3 & 5\end{bmatrix}$. Let T be the linear transformation from $\Re^n$ to $\Re^m$ defined by $\vec{x} \mapsto A\vec{x}$.

(a) (2 pts) What are the values of ~~m~~ n and ~~n~~ m? $n = 4$, $m = 2$

(b) (5 pts) What is a basis for the Null space of A. Justify.

$\begin{bmatrix}1 & 2 & 3 & 4\\1 & 2 & 3 & 5\end{bmatrix} \to \begin{bmatrix}1 & 2 & 3 & 4\\0 & 0 & 0 & 1\end{bmatrix} \Rightarrow x_4 = 0$, x_3 free x_2 free, $x_1 = -2x_2 - 3x_3 - 4x_4$

$= -2x_2 - 3x_3$

$\Rightarrow \begin{bmatrix}x_1\\x_2\\x_3\\x_4\end{bmatrix} = \begin{bmatrix}-2x_2 - 3x_3\\x_2\\x_3\\0\end{bmatrix} = x_2\begin{bmatrix}-2\\1\\0\\0\end{bmatrix} + x_3\begin{bmatrix}-3\\0\\1\\0\end{bmatrix} \Rightarrow$ Basis: $\left\{\begin{bmatrix}-2\\1\\0\\0\end{bmatrix}, \begin{bmatrix}-3\\0\\1\\0\end{bmatrix}\right\}$

(c) (5 pts) What is a basis for the Column space of A. Show your work.

Basis is pivot cols of $A \Rightarrow \left\{\begin{bmatrix}1\\1\end{bmatrix}\begin{bmatrix}4\\5\end{bmatrix}\right\}$ is a basis.

13. (5 pts) Assume $T : \Re^2 \to \Re^3$ is a linear transformation for which $T(\begin{bmatrix}1\\0\end{bmatrix}) = \begin{bmatrix}1\\2\\3\end{bmatrix}$, and $T(\begin{bmatrix}0\\1\end{bmatrix}) = \begin{bmatrix}4\\5\\6\end{bmatrix}$. What is $T(\begin{bmatrix}3\\4\end{bmatrix})$?

$$T\left(\begin{bmatrix}3\\4\end{bmatrix}\right) = T\left(3\begin{bmatrix}1\\0\end{bmatrix} + 4\begin{bmatrix}0\\1\end{bmatrix}\right) = 3\,T\begin{bmatrix}1\\0\end{bmatrix} + 4\,T\begin{bmatrix}0\\1\end{bmatrix} = 3\begin{bmatrix}1\\2\\3\end{bmatrix} + 4\begin{bmatrix}4\\5\\6\end{bmatrix}$$

$$= \begin{bmatrix}3+16\\6+20\\9+24\end{bmatrix} = \begin{bmatrix}19\\26\\33\end{bmatrix}$$

14. (5 pts) Let A be a 2×3 matrix. Consider the following subset W of $\Re^3$:

$$W = \left\{ \begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix} \in \Re^3 : A\vec{x} = \vec{0} \right\}$$

Prove that if $\vec{u}, \vec{v} \in W$, then $\vec{u} + \vec{v} \in W$.

Let $\vec{u}, \vec{v} \in W$. So $A\vec{u} = \vec{0}$ and $A\vec{v} = \vec{0}$.

$\Rightarrow A(\vec{u}+\vec{v}) = A\vec{u} + A\vec{v} = \vec{0} + \vec{0} = \vec{0}$. $\therefore \vec{u}+\vec{v} \in W$.

15. (5 pts) Let $W = \text{span}\{\vec{a}, \vec{b}\}$, where $\vec{a}, \vec{b} \in \Re^n$. Prove that $\vec{v} \in W$, $r \in \Re$ implies $r\vec{v} \in W$.

Let $\vec{v} \in W \Rightarrow \vec{v} = c_1\vec{a} + c_2\vec{b}$ with $c_1, c_2 \in \mathbb{R}$.

So $r\vec{v} = r(c_1\vec{a} + c_2\vec{b}) = (rc_1)\vec{a} + (rc_2)\vec{b} \in W$.

16. (5 pts) Assume that A and C are 5×5 matrices that satisfy $CA = I$. Prove that the equation $A\vec{x} = \vec{0}$ no solution other than $\vec{x} = \vec{0}$.

pf: If $\vec{y}$ is any sln to $A\hat{x} = \vec{0}$, then $A\vec{y} = \vec{0}$

Mult by C: $CA\vec{y} = C\vec{0} = \vec{0}$

but $CA = I \Rightarrow CA\vec{y} = \vec{y}$. $\therefore \vec{y} = \vec{0}$ //