4326 T-2 key
BP S2016

1. (6 pts) Let $\mathbb{P}_n$ be the set of polynomials of degree n or less. Let $W = \{\mathbf{p} \in \mathbb{P}_n : \mathbf{p}(0) = 1\}$. Is W a vector subspace of $\mathbb{P}_n$? Justify briefly. A formal proof is not required.

No. $1+t$ and $1-t \in W$, but $(1+t)+(1-t) = 2 \notin W$.

2. (6 pts) Let $A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 1 & 2 & 3 & 5 \end{bmatrix}$. Determine the rank and nullity of A. Explain briefly how you obtained your answers.

Row reduce $\rightarrow \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 0 & 0 & 1 \end{bmatrix}$ 2 pivots, 2 non pivot cols,

So rank$(A) = 2$ So nullity of $A = 2$.

3. Let $\vec{u} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$, and $\vec{v} = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$.

(a) (3 pts) Find a vector $\vec{w} \in \mathbb{R}^3$ for which the set $\{\vec{u}, \vec{v}, \vec{w}\}$ is linearly independent. Justify briefly.

$\vec{w} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$ $\begin{vmatrix} 1 & 1 & 0 \\ 1 & -1 & 0 \\ 0 & 0 & 1 \end{vmatrix} = -2 \neq 0$. $\therefore$ 3 cols are indep

but $\vec{x} \neq \vec{0}$, $\vec{x} \neq \vec{u}$, $\vec{x} \neq \vec{v}$

(b) (3pts) Find a vector $\vec{x} \in \mathbb{R}^3$ for which the set $\{\vec{u}, \vec{v}, \vec{x}\}$ is linearly dependent, Justify briefly.

Let $\vec{x} = \begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix}$ $\vec{x} = \vec{u} + \vec{v}$ (dependency relation.)

4. Let $\mathcal{B} = \{1 + t, t\}$. Let $\mathbb{P}_1$ be the vector space of polynomials of degree less than or equal to 1.

(a) (6 pts) Show that $\mathcal{B}$ is a basis for $\mathbb{P}_1$.

i Span: Let $p(t) \in \mathbb{P}_1$, then $p(t) = a \cdot 1 + bt$ (the most general $p \in \mathbb{P}$).

Solve $a + bt = c_1(1+t) + c_2 \cdot t \Leftrightarrow c_1 = a$ and $c_1 + c_2 = b \Rightarrow c_2 = b - a$.

Since 3 soln, span $\mathcal{B} = \mathbb{P}$.

ii $c_1(1+t) + c_2 t = 0 \Rightarrow c_1 = 0$ and $c_1 + c_2 = 0$, So $c_2 = 0$. $\therefore$ $\mathcal{B}$ is lin indep. i and ii $\Rightarrow$ $\mathcal{B}$ is a basis.

(b) (3 pts) Find the polynomial $\mathbf{r}(t)$ in $\mathbb{P}_1$ given the coördinates $[\mathbf{r}]_{\mathcal{B}} = \begin{bmatrix} 3 \\ -1 \end{bmatrix}$

$r(t) = 3(1+t) + (-1)t = 3 + 3t - t = 3 + 2t$

(c) (3 pts) What are the coordinates of the polynomial $q(t) = 2 + t$ with respect to the basis $\mathcal{B}$? (That is, what is $[q]_{\mathcal{B}}$?)

$2 + t = c_1(1+t) + c_2 t$

$\Rightarrow c_1 = 2$, $c_1 + c_2 = 1$

So $c_2 = -1$ $\therefore$ $[q]_{\mathcal{B}} = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$

5. (6 pts) Let $\vec{v}_1 = \begin{bmatrix} 1 \\ -3 \end{bmatrix}$, $\vec{v}_2 = \begin{bmatrix} 2 \\ -8 \end{bmatrix}$, $\vec{v}_3 = \begin{bmatrix} -3 \\ 7 \end{bmatrix}$. Find any two different ways to express $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ as a linear combination of $\vec{v}_1, \vec{v}_2$ and $\vec{v}_3$. Show your work.

Solve: $c_1 \begin{bmatrix} 1 \\ -3 \end{bmatrix} + c_2 \begin{bmatrix} 2 \\ -8 \end{bmatrix} + c_3 \begin{bmatrix} -3 \\ 7 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

ie, row reduce $\left[\begin{array}{ccc:c} 1 & 2 & -3 & 1 \\ -3 & -8 & 7 & 1 \end{array}\right] \to \begin{bmatrix} 1 & 2 & -3 & 1 \\ 0 & -2 & -2 & 4 \end{bmatrix} \to \left[\begin{array}{ccc:c} 1 & 2 & -3 & 1 \\ 0 & 1 & 1 & -2 \end{array}\right]$

$\begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} 5 \\ -2 \\ 0 \end{bmatrix} + t \begin{bmatrix} 5 \\ -1 \\ 1 \end{bmatrix}$

$\Rightarrow c_3$ free, $c_2 + c_3 = -2 \Rightarrow c_2 = -2 - c_3$

$c_1 + 2c_2 - 3c_3 = 1 \Rightarrow c_1 = -2c_2 + 3c_3 + 1 = -2(-2-c_3) + 3c_3 + 1$
$= 5 + 5c_3$

Let $c_3 = 0$ or $1 \Rightarrow \vec{c} = \begin{bmatrix} 5 \\ -2 \\ 0 \end{bmatrix}$ or $\vec{c} = \begin{bmatrix} 10 \\ -3 \\ 1 \end{bmatrix}$ ✓

6. (8 pts) Find the eigenvalues and basis for the corresponding eigenspaces for the matrix $A = \begin{bmatrix} 0 & 2 \\ -2 & 0 \end{bmatrix}$. Show your work.

$\det(A - \lambda I) = \begin{vmatrix} -\lambda & 2 \\ -2 & -\lambda \end{vmatrix} = \lambda^2 + 4 = 0$ if $\lambda = \pm 2i$

For $\lambda = 2i$: $(A - 2iI)\vec{v} = \begin{pmatrix} -2i & 2 \\ -2 & -2i \end{pmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow -2iv_1 + 2v_2 = 0$

or $-iv_1 + v_2 = 0$

or $v_2 = iv_1$.

$\therefore$ a basis for e-space for $\lambda = 2i$ is $\vec{v} = \begin{bmatrix} 1 \\ i \end{bmatrix}$ ✓

$\Rightarrow$ basis " " " $\lambda = -2i$: $\vec{v} = \begin{bmatrix} i \\ -i \end{bmatrix}$ ✓

7. (6 pts) Assume that A and B are both 5×5 matrices. Consider the 5×10 matrix $[A : B]$. Assume you do row operations on $[A : B]$ to convert it into $[C : I]$, where I is the 5×5 identity matrix. Express what the C matrix is in terms of A and B. Explain briefly.

Express the row ops as elementary matrices $E_1, \ldots, E_k$

$\Rightarrow [E_k \cdots E_1 A : E_k \cdots E_1 B] = [C : I]$

$(E_k \cdots E_1)B = I \Rightarrow E_k \cdots E_1 = B^{-1}$

$\therefore C = (E_k \cdots E_1)A = B^{-1}A$.

8. (6 pts) Give an example of a 2×2 matrix which has an eigenvalue of algebraic multiplicity 2, but geometric multiplicity 1. Justify briefly.

$\begin{bmatrix} 3 & 1 \\ 0 & 3 \end{bmatrix}$

$A - 3I = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$

So $(A - 3I)\vec{v} = \vec{0}$

is $\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow v_1 = 0, v_2$ free

So dim of eigen space for $\lambda = 3$ is $\underline{\underline{1}}$

which is the geometric mult

9. (6 pts) Diagonalize the matrix $A = \begin{bmatrix} 2 & 1 \\ 0 & 3 \end{bmatrix}$. (That is, find a matrix P such that $P^{-1}AP$ is diagonal.)

Find evs of A: $\lambda = 2, 3$ For $\lambda = 2$ $(A - 2I)\vec{v} = \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow v_2 = 0$

$\Rightarrow \vec{v} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

For $\lambda = 3$: $(A - 3I)\vec{v} = \begin{bmatrix} -1 & 1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow -v_1 + v_2 = 0$, or $v_2 = +v_1$

$\Rightarrow \vec{v} = \begin{bmatrix} 1 \\ +1 \end{bmatrix}$ $\therefore$ Let $P = \begin{bmatrix} 1 & 1 \\ 0 & +1 \end{bmatrix}$ ✓

10. (6 pts) Find a matrix A such that $[\vec{y}]_\mathcal{C} = A[\vec{y}]_\mathcal{B}$ for any vector $\vec{y} \in \Re^2$. (That is, find the change of basis matrix from basis $\mathcal{B}$ to basis $\mathcal{C}$, where $\mathcal{B}$ and $\mathcal{C}$ are given by $\mathcal{B} = \left\{\begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \end{bmatrix}\right\}$, and $\mathcal{C} = \left\{\begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 2 \end{bmatrix}\right\}$.

Solve $\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = b_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} + b_2 \begin{bmatrix} 1 \\ -1 \end{bmatrix} = c_1 \begin{bmatrix} 1 \\ 0 \end{bmatrix} + c_2 \begin{bmatrix} 0 \\ 2 \end{bmatrix}$, where $\begin{bmatrix} b_1 \\ b_2 \end{bmatrix} = [\vec{y}]_\mathcal{B}$

and $\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = [\vec{y}]_\mathcal{C}$

i.e, $\underbrace{\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}}_{B}\begin{bmatrix} b_1 \\ b_2 \end{bmatrix} = \underbrace{\begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}}_{C}\begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$

i.e, $B[\vec{y}]_\mathcal{B} = C[\vec{y}]_\mathcal{C} \Rightarrow [\vec{y}]_\mathcal{C} = C^{-1}B[\vec{y}]_\mathcal{B} = \begin{bmatrix} 1 & 0 \\ 0 & \frac{1}{2} \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}[\vec{y}]_\mathcal{B}$

$= \begin{bmatrix} 1 & 1 \\ \frac{1}{2} & -\frac{1}{2} \end{bmatrix}[\vec{y}]_\mathcal{B}$

11. Let $T : \mathbb{P}_2 \to \mathbb{P}_3$ be a transformation that maps a polynomial $\mathbf{p}(t)$ to the polynomial $\mathbf{p}(t) + t\mathbf{p}(t)$.

(a) (3 pts)Find $T(2 - t^2)$. $= (2 - t^2) + t(2 - t^2) = 2 - t^2 + 2t - t^3$

$= 2 + 2t - t^2 - t^3$

(b) (6 pts) Show that T is a linear transformation.

i. $T(p(t) + q(t)) = (p(t) + q(t)) + t(p(t) + q(t))$

$= (p(t) + tp(t)) + (q(t) + tq(t)) = T(p(t)) + T(q(t))$ ✓

ii $T(kp(t)) = kp(t) + t \cdot kp(t) = k(p(t) + tp(t)) = k \cdot T(p)(t)$

i and ii $\Rightarrow$ T is linear

(c) (6 pts) Find the matrix for T relative to the respective bases $\{1, t, t^2\}$, and $\{1, t, t^2, t^3\}$.

$\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \leftrightarrow 1 \mapsto 1 + t \leftrightarrow \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}$

$\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \leftrightarrow t \mapsto t + t^2 \leftrightarrow \begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \end{bmatrix}$

$\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \leftrightarrow t^2 \mapsto t^2 + t^3 \leftrightarrow \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \end{bmatrix}$ 4

$\therefore A = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$.

12. (7 pts) Assume that A, B, and P are $n \times n$ matrices, where P is invertible, and $B = P^{-1}AP$. Show that the characteristic equation of A is the same as the characteristic equation of B. You may assume that, for any two $n \times n$ matrices X and Y, $\det(XY) = \det(X)\det(Y)$.

$\det(B - \lambda I) = \det(P^{-1}AP - \lambda I) = \det(P^{-1}AP - P^{-1}(\lambda I)P) = \det(P^{-1}(A - \lambda I)P)$

$= \det P^{-1} \det(A - \lambda I) \det P = \det(A - \lambda I)$ since $P^{-1}P = I$ and $\det P^{-1} \det P = \det P^{-1}P = \det I = 1$

13. (8 pts) Let A be a $\cancel{3}$ 2×3 matrix. Consider the following subset W of $\Re^3$:

$$W = \left\{ \vec{x} \in \Re^3 : A\vec{x} = \vec{0} \right\}$$

Prove that if W is a vector subspace of V.

i. $A\vec{0} = \vec{0}$, so $\vec{0} \in W$

ii Assume $\vec{u}, \vec{v} \in W$. That is $A\vec{u} = \vec{0}$ and $A\vec{v} = \vec{0}$.

So $A(\vec{u} + \vec{v}) = A\vec{u} + A\vec{v} = \vec{0} + \vec{0} = \vec{0}$, $\therefore$ $\vec{u} + \vec{v} \in W$.

iii Assume $\vec{u} \in W$, $k \in \mathbb{R}$. Then $A\vec{u} = \vec{0}$, so

$A(k\vec{u}) = k(A\vec{u}) = k\vec{0} = \vec{0} \Rightarrow k\vec{u} \in W$.

i, ii, iii $\Rightarrow$ W is a vec subspace of V.

14. (8 pts) Assume that A is a 2×2 matrix with eigenvalues of 2 and 3. Assume $\vec{v}$ is a nonzero eigenvector for eigenvalue 2, and $\vec{w}$ is a nonzero eigenvector for the eigenvalue 3. Show that the set $\{\vec{v}, \vec{w}\}$ is linearly independent.

Assume $\{\vec{v}, \vec{w}\}$ is not indep. Then $\vec{v} = k\vec{w}$ for some $k \neq 0$, $k \in \mathbb{R}$.

$\therefore$ $A\vec{v} = 2\vec{v} = 2(k\vec{w})$

also $A\vec{v} = A(k\vec{w}) = k(A\vec{w}) = (k\,3\vec{w})$

Compare: $2(k\vec{w}) = 3k\vec{w}$

$\Rightarrow k = 0$ or $\vec{w} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$.

either contradicts $\vec{w}$ non zero or $k \neq 0$.

$\therefore$ $\{\vec{v}, \vec{w}\}$ is lin indep.