Name: *A. key*

Linear Algebra, Math 4326
Quiz 3, Spring 2016
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1. (3 pts) Assume that $\vec{v}_1, \vec{v}_2, \vec{v}_3$ are all vectors in a known vector space V. Prove that $W = \text{span}\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$ is a vector subspace of V.

Note: $\text{span}\{\vec{v}_1, \vec{v}_2, \vec{v}_3\} = \{c_1\vec{v}_1 + c_2\vec{v}_2 + c_3\vec{v}_3 : c_1, c_2, c_3 \in \mathbb{R}\}$

i $\quad 0\cdot\vec{v}_1 + 0\vec{v}_2 + 0\vec{v}_3 = \vec{0} + \vec{0} + \vec{0} = \vec{0}$, so $\vec{0} \in W$.

ii $\quad$ Let $\vec{u}, \vec{w} \in W$ So $\exists\, c_1, c_2, c_3, d_1, d_2, d_3 \in \mathbb{R}$ s.t. $\vec{u} = c_1\vec{v}_1 + c_2\vec{v}_2 + c_3\vec{v}_3$ and $\vec{w} = d_1\vec{v}_1 + d_2\vec{v}_2 + d_3\vec{v}_3$

$\Rightarrow \vec{u} + \vec{w} = (c_1\vec{v}_1 + c_2\vec{v}_2 + c_3\vec{v}_3) + (d_1\vec{v}_1 + d_2\vec{v}_2 + d_3\vec{v}_3) = (c_1+d_1)\vec{v}_1 + (c_2+d_2)\vec{v}_2 + (c_3+d_3)\vec{v}_3 \in W$

Since $c_1+d_1,\ c_2+d_2,\ c_3+d_3 \in \mathbb{R}$

iii $\quad$ Let $\vec{u} \in W$, $c \in \mathbb{R}$. Then $\exists\, c_1, c_2, c_3 \in \mathbb{R}$ s.t. $\vec{u} = c_1\vec{v}_1 + c_2\vec{v}_2 + c_3\vec{v}_3$.

$\Rightarrow c\vec{u} = c\,(c_1\vec{v}_1 + c_2\vec{v}_2 + c_3\vec{v}_3) = (c\,c_1)\vec{v}_1 + (c\,c_2)\vec{v}_2 + (c\,c_3)\vec{v}_3 \in W$

Since $cc_1, cc_2\ cc_3 \in \mathbb{R}$.

i, ii, and iii $\Rightarrow$ W is a vector subspace of V.

2. (2pts) Consider the vector space $\Re^2$. Both $\mathcal{B}$ and $\mathcal{C}$ are bases for $\Re^2$, where $\mathcal{B} = \left\{\begin{bmatrix}1\\1\end{bmatrix}, \begin{bmatrix}1\\-1\end{bmatrix}\right\}$, and $\mathcal{C} = \left\{\begin{bmatrix}1\\0\end{bmatrix}, \begin{bmatrix}1\\2\end{bmatrix}\right\}$. Let $\vec{x} = \begin{bmatrix}1\\1\end{bmatrix}$.

(a) Find the coordinates of $\vec{x}$ with respect to each of the two bases. That is, find $[\vec{x}]_{\mathcal{B}}$, and $[\vec{x}]_{\mathcal{C}}$.

Solve $c_1\begin{bmatrix}1\\1\end{bmatrix} + c_2\begin{bmatrix}1\\-1\end{bmatrix} = \begin{bmatrix}1\\1\end{bmatrix}$. Then $[X]_{\mathcal{B}} = \begin{bmatrix}1\\1\end{bmatrix}_{\mathcal{B}} = \begin{bmatrix}c_1\\c_2\end{bmatrix}$.

By row reduction: $\left[\begin{array}{cc|c}1 & 1 & 1\\1 & -1 & 1\end{array}\right] \to \left[\begin{array}{cc|c}1 & 1 & 1\\0 & -2 & 0\end{array}\right] \Rightarrow c_2 = 0,\ c_1 = 1$ So $[X]_{\mathcal{B}} = \begin{bmatrix}1\\0\end{bmatrix}$

Similarly, for $[X]_{\mathcal{C}}$, row reduce $\left[\begin{array}{cc|c}1 & 1 & 1\\0 & 2 & 1\end{array}\right] \Rightarrow 2c_2 = 1$ or $c_2 = \frac{1}{2}$ and $c_1 + c_2 = 1 \Rightarrow c_1 = \frac{1}{2}$, so $[X]_{\mathcal{C}} = \begin{bmatrix}\frac{1}{2}\\\frac{1}{2}\end{bmatrix}$

Note: $[X]_{\mathcal{B}}$ is clear "by inspection." $[X]_{\mathcal{C}}$ can also be done by inspection, but it is not quite so obvious.

(b) Extra credit (3pts). Find a matrix A such that $[\vec{y}]_{\mathcal{C}} = A[\vec{y}]_{\mathcal{B}}$ for any vector $\vec{y} \in \Re^2$. Write $\vec{y}$ as a linear combination of basis vectors in both $\mathcal{B}$ and $\mathcal{C}$:

$\vec{y} = b_1\begin{bmatrix}1\\1\end{bmatrix} + b_2\begin{bmatrix}1\\-1\end{bmatrix} = d_1\begin{bmatrix}1\\0\end{bmatrix} + d_2\begin{bmatrix}1\\2\end{bmatrix}$ where $\begin{bmatrix}b_1\\b_2\end{bmatrix} = [\vec{y}]_{\mathcal{B}}$ and $\begin{bmatrix}d_1\\d_2\end{bmatrix} = [\vec{y}]_{\mathcal{C}}$

rewrite: $\vec{y} = \begin{bmatrix}1 & 1\\1 & -1\end{bmatrix}\begin{bmatrix}b_1\\b_2\end{bmatrix} = \begin{bmatrix}1 & 1\\0 & 2\end{bmatrix}\begin{bmatrix}d_1\\d_2\end{bmatrix}$. Let $\mathcal{B} = \begin{bmatrix}1 & 1\\1 & -1\end{bmatrix}$, $\mathcal{C} = \begin{bmatrix}1 & 1\\0 & 2\end{bmatrix}$.

Then $\vec{y} = \mathcal{B}\begin{bmatrix}b_1\\b_2\end{bmatrix} = \mathcal{C}\begin{bmatrix}d_1\\d_2\end{bmatrix}$. Solve for $\begin{bmatrix}d_1\\d_2\end{bmatrix} = \mathcal{C}^{-1}\mathcal{B}\begin{bmatrix}b_1\\b_2\end{bmatrix}$

ie, $[\vec{y}]_{\mathcal{C}} = \mathcal{C}^{-1}\mathcal{B}\,[\vec{y}]_{\mathcal{B}}$, so $A = \mathcal{C}^{-1}\mathcal{B} = \frac{1}{2}\begin{bmatrix}2 & -1\\0 & 1\end{bmatrix}\begin{bmatrix}1 & 1\\1 & -1\end{bmatrix} = \begin{bmatrix}\frac{1}{2} & \frac{3}{2}\\\frac{1}{2} & -\frac{1}{2}\end{bmatrix}$