Name: A.K.

# Linear Algebra, Math 4326
Quiz 7, Spring 2016
B. Peckham

1. (1 pt) Let $D$ and $P$ be $n \times n$ matrices where $D$ is diagonal. Show that $PDP^T$ is symmetric.

$$(PDP^T)^T = \underbrace{P^{TT} D^T P^T}_{\text{by transpose properties}} = \underbrace{P D^T P^T}_{P^{TT}=P} = \underbrace{PDP^T}_{D^T = D \text{ since } D \text{ is diagonal}}$$

$\therefore PDP^T$ is symmetric

2. (3 pts) Let $A$ be a symmetric $n \times n$ matrix with $\lambda_1$ and $\lambda_2$ two unequal eigenvalues and $\vec{v}_1$ and $\vec{v}_2$ corresponding eigenvectors. Show that $\vec{v}_1$ and $\vec{v}_2$ are orthogonal.

$$\lambda_1(\vec{v}_1 \cdot \vec{v}_2) = (\lambda_1 \vec{v}_1)\cdot\vec{v}_2 = (A\vec{v}_1)\cdot\vec{v}_2 = (A\vec{v}_1)^T\vec{v}_2 = \vec{v}_1^T A \vec{v}_2 = \vec{v}_1^T(\lambda_2\vec{v}_2) = \lambda_2 \vec{v}_1^T\vec{v}_2 = \lambda_2 \vec{v}_1\cdot\vec{v}_2$$

$\Rightarrow (\lambda_1 - \lambda_2)\vec{v}_1\cdot\vec{v}_2 = 0$. $\lambda_1 \neq \lambda_2 \Rightarrow \lambda_1 - \lambda_2 \neq 0$, so $\vec{v}_1 \cdot \vec{v}_2 = 0$

ie, $\vec{v}_1$ and $\vec{v}_2$ are orthogonal

3. Consider the quadratic form $Q(\vec{x}) = x_1^2 + 2\sqrt{6}x_1x_2 + 2x_2^2$.

   (a) (1pt) Find a symmetric matrix $A$ for which $Q(\vec{x}) = \vec{x}^T A \vec{x}$.

$$A = \begin{bmatrix} 1 & \sqrt{6} \\ \sqrt{6} & 2 \end{bmatrix}$$

   (b) (EC 3pts) Find an orthogonal matrix $P$ such that $\vec{x} = P\vec{y}$ implies that $Q(P\vec{y}) = ay_1^2 + by_2^2$ for some $a$ and $b$. Determine $a$ and $b$. Hint: one eigenvector for $\begin{bmatrix} 1 & \sqrt{6} \\ \sqrt{6} & 2 \end{bmatrix}$ is $\begin{bmatrix} \sqrt{6} \\ 3 \end{bmatrix}$.

Want to "orthogonally diagonalize" $A$. Want $P$ with columns unit eigenvectors of $A$. One choice: $\vec{p}_1 = \dfrac{\begin{bmatrix} \sqrt{6} \\ 3 \end{bmatrix}}{\left\| \begin{bmatrix} \sqrt{6} \\ 3 \end{bmatrix} \right\|} = \dfrac{1}{\sqrt{15}}\begin{bmatrix} \sqrt{6} \\ 3 \end{bmatrix}$

For the second eigenvector, by inspection (since it must be orthog. to $\vec{p}_1$) $\vec{p}_2 = \dfrac{1}{\sqrt{15}}\begin{bmatrix} 3 \\ -\sqrt{6} \end{bmatrix}$. Check: $\vec{p}_1 \cdot \vec{p}_2 = \dfrac{3\sqrt{6}}{\sqrt{15}^2} - \dfrac{3\sqrt{6}}{\sqrt{15}^2} = 0$ ✓

$\therefore$ Let $P = [\vec{p}_1\ \vec{p}_2] = \dfrac{1}{\sqrt{15}}\begin{bmatrix} \sqrt{6} & 3 \\ 3 & -\sqrt{6} \end{bmatrix} \Rightarrow P^{-1} = P^T = \dfrac{1}{\sqrt{15}}\begin{bmatrix} \sqrt{6} & 3 \\ 3 & -\sqrt{6} \end{bmatrix}$,

$$P^TAP = \frac{1}{15}\begin{bmatrix} \sqrt{6} & 3 \\ 3 & -\sqrt{6} \end{bmatrix}\begin{bmatrix} 1 & \sqrt{6} \\ \sqrt{6} & 2 \end{bmatrix}\begin{bmatrix} \sqrt{6} & 3 \\ 3 & -\sqrt{6} \end{bmatrix} = \frac{1}{15}\begin{bmatrix} \sqrt{6} & 3 \\ 3 & -\sqrt{6} \end{bmatrix}\begin{bmatrix} 4\sqrt{6} & -3 \\ 12 & \sqrt{6} \end{bmatrix} = \frac{1}{15}\begin{bmatrix} 60 & 0 \\ 0 & -15 \end{bmatrix} = \begin{bmatrix} 4 & 0 \\ 0 & -1 \end{bmatrix} \Rightarrow a = 4,\ b = -1$$