MIDTERM 1 (Thursday, February 11 2016, 5-6:30 in MonH 70)
TOPIC LIST
(as of February 9, 2016- not yet final)
Linear Algebra, Math 4326
Bruce Peckham

1. Definitions and Notation
   (a) Solution set for a system of linear equations; parametric form for infinite solution sets
   (b) Coefficient matrix and augmented matrix for a system of linear equations
   (c) Basic variables versus free variables in a system of equations
   (d) Pivot elements and pivot columns of a matrix
   (e) Row echelon form, reduced row echelon form
   (f) Vector space/subspace
   (g) Linear combinations of vectors in $\Re^n$
   (h) Matrix multiplication: as a linear combination of column vectors, or as row-column combinations
   (i) Linearly independent/dependent subsets of $\Re^n$
   (j) A dependency relation for a set of vectors
   (k) Span of a set of vectors
   (l) Basis of $\Re^n$ or a subspace of $\Re^n$
   (m) Dimension of $\Re^n$ or a subspace of $\Re^n$
   (n) Inverse of a square matrix
   (o) Transpose of a matrix
   (p) Linear transformation
   (q) Matrix of a linear transformation
   (r) A linear transformation being *one-to-one* and/or *onto*
   (s) Determinant of any $(n+1) \times (n+1)$ matrix in terms of determinants of $n \times n$ matrices (expand along any row or column)

2. Examples, results and constructions to know:
   (a) Row reduction to solve a system of linear equations
   (b) Examples of systems of equations with no solutions, unique solutions, infinity of solutions (k-dimensional sets of solutions for any k)
   (c) Geometric interpretation of linear or affine subsets of $\Re^n$ (point, line, plane, hyperplane)
   (d) Setting up equations to solve to determine whether a set of vectors in $\Re^n$ is linearly independent. Know how to interpret the solution to your system.
   (e) Setting up equations to solve to determine whether a set of vectors in $\Re^n$ spans $\Re^n$. Know how to interpret the solution to your system.
   (f) Know how to compute an inverse of a square matrix using row reduction of $[A : I]$.
   (g) Know how to set up equations to determine whether a linear transformation is one-to-one and/or onto.
   (h) Compute a determinant of a square matrix using row reduction
   (i) For any elementary row operation on a matrix A, find an elementary matrix E for which multiplication on the left (EA) gives the same results as the row operation.

(j) Write the inverse of a matrix A as a product of elementary matrices. Do the same for A.

(k) Find a basis for the Col(A) and/or Null(A) for a given $m \times n$ matrix.

3. Proofs to know:

(a) Show that if $W_1 = \text{Span}\{\vec{\text{v}}_1, ..., \vec{\text{v}}_\text{k}\}$ is linearly dependent in $\Re^n$, and $\vec{v}_{k+1}$ is any vector in $\Re^n$, then $W_2 = \text{Span}\{\vec{\text{v}}_1, ..., \vec{\text{v}}_\text{k}, \vec{\text{v}}_{\text{k+1}}\}$ is also linearly dependent. (Proof: If W_1 is linearly dependent, then there are constants c_i such that $c_1\vec{v}_1 + ... + c_k\vec{v}_k = \vec{0}$, with not all weights c_i equal to zero. This implies that $c_1\vec{v}_1 + ... + c_k\vec{v}_k + 0\vec{v}_{k+1} = \vec{0}$, with not all weights c_i equal to zero. So W_2 is linearly dependent as well.)

(b) Show that if a vector is removed from a linearly independent set with more than one vector, then the remaining set is still linearly independent. (Proof: This is the contrapositive of the preceding claim, so the same proof works.)

(c) Show that if a set W of vectors is linearly dependent, then at least one of the vectors can be solved for in terms of the rest of the vectors in the set. Show the converse is also true. (Proof: If W is linearly dependent, then there are constants c_j, not all zero for which $c_1\vec{v}_1 + ... + c_k\vec{v}_k = \vec{0}$. Assume i is an index for which $c_i \neq 0$. Then you can solve for $\vec{v}_i = -\frac{c_1}{c_i}\vec{v}_1 - ... - \frac{c_{i-1}}{c_i}\vec{v}_{i-1} - \frac{c_{i+1}}{c_i}\vec{v}_{i+1} - ... - \frac{c_k}{c_i}\vec{v}_k$.) (Proof of the converse: Assume $\vec{v}_i = c_1\vec{v}_1 + ... + c_{i-1}\vec{v}_{i-1} + c_{i+1}\vec{v}_{i+1} + ... + c_k\vec{v}_k$. Now move all terms to the left to get a linear combination of all k vectors which yields $\vec{0}$. Indicate why not all weights in this linear combination are not all zero.)

(d) $W = \text{Span}\{\vec{\text{v}}_1, ..., \vec{\text{v}}_\text{k}\}$ is a vector subspace of $\Re^n$, where each $\vec{v}_i \in \Re^n$. (Proof: See Example 1, Sec. 2.8, p. 198 for the proof with $k = 2$. Example 3 suggests the proof for arbitrary k.)

(e) The solutions to $A\vec{x} = \vec{0}$ is a vector subspace of $\Re^n$, where A is an $m \times n$ matrix. (See Theorem 12, Sec. 2.8, p. 150.)

(f) The column space of A is a vector subspace of $\Re^m$, where A is an $m \times n$ matrix. (Since the column space is by definition the span of the columns of A, the proof is the same as the item (d).)

(g) If A, L, R are all $n \times n$ matrices, and $LA = I$ and $AR = I$, then $L = R$. (Proof sketch: Consider the product LAR. Since $LA = I$, this reduces to R. Since $AR = I$, this also reduces to L. Therefore $L = R$.

(h) IMT: (a) implies (g). (Proof sketch: Assume A is invertible. Plug $A^{-1}\vec{b}$ in for $\vec{x}$ in the equation $A\vec{x} = \vec{b}$ to show that the equation is satisfied.)

(i) IMT: (j) implies (d). (Proof: Given: $CA = I$. Assume $\vec{y}$ is any solution to $A\vec{x} = \vec{0}$. That is, $A\vec{y} = \vec{0}$. Multiply both sides on the left by matrix C: $CA\vec{y} = C\vec{0} = \vec{0}$. But $CA\vec{y} = I\vec{y} = \vec{y}$. So $\vec{y} = \vec{0}$.)

(j) IMT: (g) implies (k). (Proof: Assume $A\vec{x} = \vec{b}$ has a solution for all $\vec{b}$. Let $\vec{u}$ be the solution for $A\vec{x} = (1, 0)^t$, and $\vec{v}$ be the solution for $A\vec{x} = (0, 1)^t$. Then if D is the matrix $[\vec{u}\vec{v}]$, $AD = I$.)

4. (Parts of) HW problems might be asked on the test.

5. Anything else we've covered that I think is easy.