MIDTERM 2 (Tuesday, March 22 2016, 5-6:30pm in Chem 200)
TOPIC LIST
(as of March 22, 2016Linear Algebra, Math 4326
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Test 2 will cover sections 4.1-4.7 and 5.1-5.5 in the text. From Sec. 5.5, you will only be asked how to computed eigenvalues and eigenvectors for a matrix which has at least one (and therefore at least two) complex eigenvalue(s).

1. Definitions and Notation
   (a) Abstract vector space (not just $\Re^n$) and/or subspace of V
   (b) A linear combination of a set of vectors
   (c) The set of all linear combinations of a set of vectors (that is, the span of the set of vectors)
   (d) Linearly independent/dependent sets of vectors
   (e) A dependency relation for a set of vectors
   (f) A basis of a vector space or a subspace
   (g) The standard basis for $\Re^n$
   (h) The dimension of a vector space or subspace
   (i) Linear transformation
   (j) Null space of a matrix
   (k) Column space of a matrix
   (l) Kernel and range of a linear transformation (Note: when the linear transformation is between $\Re^n$ and $\Re^m$ instead of beetween two abstract vector spaces, the transformation must be multiplication by some matrix A; in this case the kernel is the null space of A, and the range is the column space of A.)
   (m) Matrix of a linear transformation from $\Re^n$ to $\Re^m$.
   (n) A linear transformation being *one-to-one* and/or *onto*
   (o) Coordinates with respect to a basis of a vector in a vector space
   (p) The change of coordinates matrix from a basis $\mathcal{B}$ to the standard basis for $\Re^n$
   (q) The change of coordinates matrix from one basis to another basis (in any vector space)
   (r) Rank of a matrix
   (s) Eigenvalues and eigenvectors of a square matrix
   (t) Algebraic multiplicity (the multiplicity of the root in the characteristic equation) and geometric multiplicity (the dimension of the eigenspace corresponding to the eigenvalue) of an eigenvalue of a matrix; note that $1 \leq$ geometric multiplicity $\leq$ algebraic multiplicity for each eigenvalue of a matrix. This not too hard to see when A is diagonal because when you solve $(A - \lambda I)\vec{v} = \vec{0}$, all entries of $\vec{v}$ which are do not correspond to eigenvalue λ must be zero. This leaves the maximum number of free variables (the geometric multiplicity) less than or equal to the number of times λ appears on the diagonal (the algebraic multiplicity).
   (u) Similar matrices
   (v) Diagonalizable matrix

2. Examples, results and constructions to know:

   (a) Row reduction to solve a system of linear equations (as for Test 1)

   (b) Geometric interpretation of subspaces of $\Re^n$ (point, line, plane, hyperplane - all passing through the origin)

   (c) Setting up equations to solve to determine whether a set of vectors in $\Re^n$ is linearly independent. Know how to interpret the solution to your system.

   (d) Setting up equations to solve to determine whether a set of vectors in $\Re^n$ spans $\Re^n$. Know how to interpret the solution to your system.

   (e) Know how to compute the inverse of a square matrix (if it is invertible) using row reduction of $[A : I]$.

   (f) Know how to set up equations to determine whether a linear transformation is one-to-one and/or onto.

   (g) Compute a determinant of a square matrix using row reduction or expanding along a row or column

   (h) For any elementary row operation on a matrix A, find an elementary matrix E for which multiplication on the left (EA) gives the same results as the row operation.

   (i) Write the inverse of a matrix A as a product of elementary matrices. Do the same for A.

   (j) Find a basis for the Col(A) and/or Null(A) for a given $m \times n$ matrix.

   (k) Determine the rank of a matrix (the dimension of its row space or the dimension of its column space)

   (l) Find a change of basis matrix for coordinates with respect to two different bases.

   (m) Compute eigenvalues and eigenvectors for a given square matrix; find bases for the eigenspaces.

   (n) Find a change of coordinates to diagonalize a matrix. (That is, find a matrix P such that PAP^{-1} is diagonal.)

   (o) Give an example of a matrix which is not diagonalizable. (Note: it must have at least one repeated eigenvalue.)

   (p) Give an example of a matrix with repeated eigenvalues which IS diagonalizable.

   (q) If V and W are vector spaces, $T : V \to W$ is a linear transformation, and you are given $T(\vec{v})$ for all $\vec{v}$ in a basis for V, determine $T(\vec{v})$ for *any* $\vec{v} \in V$. Special case: If V and W are both $\Re^2$, $\vec{v_1}$ and $\vec{v_2}$ are linearly independent vectors in $\Re^2$, and $\vec{x} \mapsto A\vec{x}$, where A is an unknown 2×2 matrix; if you are given $A\vec{v_1}$ and $A\vec{v_2}$, then determine A.

3. Proofs to know:

   (a) Let $\vec{v_1}, ..., \vec{v_n}$ be vectors in V. Show that $\text{Span}\{\vec{v_1}, ..., \vec{v_n}\}$ is a vector subspace of V. (Proof: See the solution to problem 1 in quiz 3.)

   (b) The nullspace of an $m \times n$ matrix is a subspace of $\Re^n$. (Proof: see Theorem 12, Sec. 2.8, p. 150.)

   (c) The set of eigenvectors corresponding to a single eigenvalue of an $n \times n$ matrix (together with the zero vector) is a vector subspace of $\Re^n$. (OK to assume that the nullspace of an $n \times n$ matrix is a vector subspace of $\Re^n$.) Proof hint: Show that the eigenvectors corresponding to λ satisfy $(A - \lambda I)\vec{v} = \vec{0}$.

   (d) Two eigenvectors corresponding to different eigenvalues form a linearly independent set, or a corollary: if A is a 2×2 matrix with distinct real eigenvalues, then $\Re^2$ has a basis consisting of eigenvectors of A. (Proof sketch: Show the two eigenvectors being linearly dependent leads to a the corresponding eigenvalues being equal, which is a contradiction. Use the fact that eigenvectors are not the zero vector. For two vectors, both being nonzero and being linearly dependent means that each one is a nonzero multiple of the other. So $\vec{v_1} = k\vec{v_2}$ with $k \neq 0, \vec{v_1} \neq \vec{0}$, and $\vec{v_2} \neq \vec{0}$. Being eigenvectors implies that $A\vec{v_1} = \lambda_1\vec{v_1}$, and $A\vec{v_2} = \lambda_2\vec{v_2}$. So $A\vec{v_1} = \lambda_1\vec{v_1} = A(k\vec{v_2}) = kA\vec{v_2} = k\lambda_2\vec{v_2} = \lambda_2\vec{v_1}$. Comparing the second and last expressions shows that $\lambda_1 = \lambda_2$, which is a contradiction.)

   (e) Show that, if $\{\vec{v_1}, ..., \vec{v_n}\}$ is a basis for a vector space V, then any $\vec{v} \in V$ *has* a *unique* set of real numbers $c_1, c_2, ..., c_n$ such that $\vec{v} = c_1\vec{v_1} + ... + c_n\vec{v_n}$. (Proof: the existence of the constants comes directly from the fact that a basis of V spans V; the uniqueness comes from the fact that the basis vectors are linearly independent: show that if there were two different sets of constants which produced the same vector, then this would lead to an equation that would show that the vectors were linearly dependent. This would contradict the basis vectors being linearly independent.)

   (f) Show that similar matrices have the same characteristic equation (and therefore the same eigenvalues). (Proof: Theorem 4, Sec. 5.2 p. 279. The key fact it uses is that the determinant of a product of matrices is the product of the determinants.)

4. (Parts of) problems similar to HW problems assigned for Chapters 4-5 might be asked on the test. This includes problems assigned but not to be turned in.

5. Quiz questions from chapters 4 and 5.

6. Anything else we've covered that I think is easy.