MIDTERM 3 (Thursday, April 21 2016, 5:15-6:45pm in MWAH 195)
TOPIC LIST
(as of April 18, 2016)
Linear Algebra, Math 4326
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Test 3 will cover sections 5.5, 6.1-6.5, 6.7, 7.1, 7.2, 7.4 in the text. Excluded topics are listed along with the homework assignments on the course webpage.

1. Definitions and Notation

   (a) Dot product in $\mathbb{R}^n : \vec{a} \bullet \vec{b} = \vec{a}^T \vec{b}$
   (b) Length of a vector: $||\vec{a}|| = \sqrt{\vec{a} \bullet \vec{a}}$; unit vector
   (c) Distance from vector $\vec{a}$ to $\vec{b}$: $||\vec{a} - \vec{b}||$
   (d) Orthogonal vectors ($\vec{a}$ is orthogonal to $\vec{b}$ if $\vec{a} \bullet \vec{b} = 0$)
   (e) Orthogonal set of vectors (each vector in the set is orthogonal to every other vector in the set)
   (f) Orthonormal set (orthogonal set and each vector has unit length)
   (g) Orthogonal/orthonormal basis for a vector space or subspace
   (h) The orthogonal complement to a subspace $W \subset V$: $W^\perp = \{\vec{v} \in V : \vec{v} \bullet \vec{w} = 0 \ \forall \vec{w} \in W\}$
   (i) Angle between two vectors (defined by $\vec{a} \bullet \vec{b} = ||\vec{a}|| ||\vec{b}|| \cos(\theta)$)
   (j) The projection of a vector onto a subspace
   (k) Linear transformation
   (l) Null space of a matrix
   (m) Column space of a matrix
   (n) Matrix of a linear transformation from $\mathbb{R}^n$ to $\mathbb{R}^m$.
   (o) Eigenvalues and eigenvectors of a square matrix
   (p) Orthogonal matrix ($A^{-1} = A^T$, that is, columns (and therefore also rows) of A form an orthonormal set)
   (q) Diagonal matrix; diagonalizable matrix; orthogonally diagonalizable matrix

2. Examples, results and constructions to know:

   (a) Compute (real and/or complex) eigenvalues and eigenvectors for a given square matrix; find bases for the eigenspaces.
   (b) Find a change of coordinates to diagonalize a matrix. That is, find a matrix P such that $P^{-1}AP$ is diagonal. (Choose the columns of P to be eigenvectors of A.)
   (c) Find a change of coordinates to convert a 2×2 matrix with complex eigenvalues and eigenvectors into the "canonical" form. That is, find a matrix P such that $P^{-1}AP = \begin{bmatrix} a & -b \\ b & a \end{bmatrix}$. (Choose the columns of P to be the real and imaginary parts of any complex eigenvector of A.)
   (d) Orthogonally diagonalize a given symmetric matrix. (That is, find an orthogonal matrix P such that $P^T AP$ is diagonal.)
   (e) Give an example of a matrix which is not diagonalizable. (Note: it must have at least one repeated eigenvalue.)
   (f) Give an example of a matrix with repeated eigenvalues which IS diagonalizable.
   (g) If V and W are vector spaces, $T : V \to W$ is a linear transformation, and you are given $T(\vec{v})$ for all $\vec{v}$ in a basis for V, determine $T(\vec{v})$ for *any* $\vec{v} \in V$. Special case: If V and W are both $\mathbb{R}^2$, $\vec{v_1}$ and $\vec{v_2}$ are linearly independent vectors in $\mathbb{R}^2$, and $\vec{x} \mapsto A\vec{x}$, where A is an unknown 2×2 matrix; if you are given $A\vec{v_1}$ and $A\vec{v_2}$, then determine A.

(h) Compute length of a vector, dot products of vectors, inner products of (abstract) vectors, angles between vectors using dot/inner products

(i) Determine whether a given set of vectors is orthogonal/orthonormal.

(j) Compute an orthogonal or orthonormal basis from a given basis (Gram-Schmidt)

(k) Compute the projection of a given vector onto another vector, or onto a subspace spanned by a set of vectors (if the spanning set is not orthogonal, first create an orthogonal spanning set using Gram-Schmidt); decomposition of a vector into a projection onto a subspace W and its orthogonal complement $W^\perp$ (notation: $\vec{y} = \hat{y} + \vec{z}$, where $\hat{y} \in W$ and $\vec{z} \in W^\perp$).

(l) Find a least squares solution to an inconsistent system $A\vec{x} = \vec{b}$ and the least squares error (the distance from $A\hat{x}$ to $\vec{b}$) using one or both of these methods:

   i. Let $W = col(A)$, decompose $\vec{b} = \hat{b} + \vec{z}$ where $\hat{b} \in W, z \in W^\perp$, solve $A\vec{x} = \hat{b}$ for $\hat{x}$.

   ii. Solve $A^T A\vec{x} = A^T\vec{b}$ for $\hat{x}$; then $\hat{b} = A\hat{x}$.

(m) Rewrite a quadratic form as $Q(\vec{x}) = \vec{x}^T A\vec{x}$. Find a change of coordinates $\vec{x} = P\vec{y}$ to eliminate "mixed" terms. (That is orthogonally diagonalize the matrix of the quadratic form.)

(n) Find the singular values of a given $m \times n$ matrix (the square root of the eigenvalues of $A^T A$).

(o) Find the maximum of $||A\vec{x}||$ for $\vec{x}$ with restricted have norm 1 (the maximum singular value)

(p) Given a matrix A, find the vector $\vec{x}$ with norm 1 that maximizes $||A\vec{x}||$. (This is the unit eigenvector of $A^T A$ corresponding to its maximum eigenvalue.)

3. Proofs to know:

   (a) The law of cosines (assuming the Pythagorean theorem)

   (b) $\vec{a} \bullet \vec{b} = ||\vec{a}||||\vec{b}|| \cos(\theta)$ (assuming the law of cosines)

   (c) If W is a vector subspace of V, then so is $W^\perp$. (Prob. 30, sec. 6.1 or Quiz 6 solutions on course webpage)

   (d) If $W = span\{\vec{v}_1, ..., \vec{v}_p\}$, and $\vec{v}$ is orthogonal to every vector $\vec{v}_i$, then $\vec{v}$ is orthogonal to every vector in W.

   (e) $Col(A)^\perp = Nul(A^T)$. (Theorem 3, Sec. 6.1)

   (f) The columns of the matrix U are orthogonal if and only if $U^T U = I$.

   (g) If U is an orthogonal matrix, then $||U\vec{x}|| = ||\vec{x}||$.

   (h) If A is a symmetric matrix, and $\vec{v}_1$ and $\vec{v}_2$ are eigenvectors corresponding to *different* eigenvalues, then $\vec{v}_1$ and $\vec{v}_2$ are orthogonal. (Theorem 1, Sec. 7.1, p. 397-8)

   (i) $A^T A$ is symmetric.

   (j) If a square matrix A is orthogonally diagonalizable, then it is symmetric. (That is, if $A = P^T DP$, where $P^T P = I$, then $A^T = A$.) See the bottom of p. 398, just before Theorem 2.

   (k) The eigenvalues of $A^T A$ are real and nonnegative. Pf: See the top of p. 418, Section 7.4.

   (l) If $Q(\vec{x}) = \vec{x}^T A\vec{x}$, and $\vec{x} = P\vec{y}$, then $Q(\vec{x}) = \vec{y}^T P^T AP\vec{y}$. (NOTE: this becomes especially important when A is symmetric because then P can be chosen to be orthogonal and $P^T AP$ is a diagonal matrix.)

   (m) (possible Extra Credit question only) Consider the system $A\vec{x} = \vec{b}$ which is assumed to be inconsistent. Derive the "normal equations" $A^T A\vec{x} = A^T\vec{b}$ for $\hat{x}$ assuming $A\hat{x} = \hat{b}$ where $\hat{b}$ is the orthogonal projection of $\vec{b}$ onto the $Col(A)$.

4. (Parts of) problems similar to HW problems assigned for Sec. 5.5 and Chapters 6-7 might be asked on the test. This includes problems assigned but not to be turned in.

5. Quiz questions from chapters 6 and 7.

6. Anything else we've covered that I think is easy.