MIDTERM 1 (Monday, October 20, 2008; 3-4:30 in LSBE 129)
TOPIC LIST
(as of October 16, 2008)
Real Variables, Math 5201
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1. Definitions

   (a) Operations on sets: $A \cup B, A \cap B, A - B, A^c$
   (b) $A \times B$
   (c) Function, domain, codomain, image
   (d) A is a dense subset of B (in a metric space)
   (e) $f(A), f^{-1}(A)$ (Image and inverse image of A)
   (f) χ_A (or 1_A, the indicator function of A)
   (g) $A \sim B$ (A is "equinumerous" with B, or A has the same cardinality as B: $\exists f : A \to B$ that is one-to-one and onto)
   (h) Countable ($A \sim N$, where N is the set of natural numbers, or positive integers), uncountable.
   (i) Definition of the Reals as the collection of all "cuts" of the rationals. Consequence (lub property). (with "appropriate identifications").
   (j) Metric Space, subspace, product space
   (k) Complete metric space (every Cauchy sequence converges)
   (l) Specific metrics: Euclidean, max, sum, discrete
   (m) Topological Space, subspace, product space
   (n) Toplogy: open, closed, limits of A, accumulation (cluster) points of A, $\overline{A}, A^\circ, \partial A$
   (o) homeomorphism
   (p) For a set A in a metric space: bounded, unbounded
   (q) For a bounded set $A \in \Re$: least upper bound, greatest lower bound
   (r) Least upper bound principle in $\Re$: any set bounded above has a least upper bound
   (s) Sequence, subsequence, bounded sequence, monotone sequence in a metric space
   (t) ϵ, N definition of $s_n \to L$ in a metric space.
   (u) In $\Re$ with the Euclidean metric: $\limsup\{s_n\}_{n=1}^{\infty}$ and $\liminf\{s_n\}_{n=1}^{\infty}$, including the definitions of the M_n's and m_n's which are the l.u.b.'s and g.l.b.'s, respectively, for $T_n = \{s_n, s_{n+1}, ...\}$ (when the T_n's are bounded above and/or below).
   (v) Definition for a sequence to be Cauchy (ϵ, N)

(w) Continuity of $f : M_1 \to M_2$ at a:

i. ϵ, δ

ii. Inverse image of open ball around $f(a)$ contains an open ball around a.

iii. $x_n \to a \Rightarrow f(x_n) \to f(a)$

(x) Continuity of $f : M_1 \to M_2$ on M_1:

i. Continuous at each $a \in M_1$ using any of the above notions of continuous at a point.

ii. Inverse image of any open set is open

iii. Inverse image of any closed set is closed

(y) Uniform continuity of $f : M_1 \to M_2$ on M_1. (ϵ-δ)

(z) $A \subset M$ is (sequentially) compact.

2. Theorems/results to know (including their proofs):

(a) Relationships between sets:

i. DeMorgan's laws: $(A \cup B)^c = A^c \cap B^c$; $(A \cap B)^c = A^c \cup B^c$

ii. $f^{-1}(A \cup B) = f^{-1}(A) \cup f^{-1}(B)$

iii. $f^{-1}(A \cap B) = f^{-1}(A) \cap f^{-1}(B)$

iv. $f(A \cup B) = f(A) \cup f(B)$

v. $f(A \cap B) \subseteq f(A) \cap f(B)$ Know an example showing strict inclusion.

vi. $f(A) - f(B) \subseteq f(A - B)$ Know an example showing strict inclusion.

(b) Cardinality equivalences and inequivalences.

i. $N \sim 2I$. (The natural numbers are equivalent to the even natural numbers.) Explicit construction.

ii. $N \sim Q$. (The natural numbers are equivalent to the rational numbers.) Explicit construction.

iii. $N \not\sim [0, 1]$ (Proof by contradiction.)

iv. $\sqrt{2}$ is irrational

(c) Metric Space results:

i. Properties of open and closed sets, for example A_α open $\Rightarrow \cup A_\alpha$, $A_1 \cap A_2$ both are open. Counterexamples (eg., the union of closed sets which is not closed)

ii. Show A open in $M \Leftrightarrow A^c$ closed in M.

iii. Uniqueness of limit of a convergent sequence (proof)

iv. Bolzano-Weierstrass Theorem: Any bounded sequence has a convergent subsequence

v. $s_n \to L \Rightarrow \{s_n\}$ is Cauchy. (ϵ, N proof)

vi. $\{s_n\}$ is Cauchy $\Rightarrow \{s_n\}$ converges. (Idea of proof only: Cauchy implies bounded. Bounded implies $\exists$ a convergent subsequence (by Bolzano-Weierstrauss). The Cauchy assumption now implies that the original must converge to the same limit as the subsequence. More details for extra credit only.)

vii. A compact $\Rightarrow A$ is closed and bounded.

viii. $A \subset M$ compact and $B \subset N$ compact $\Leftrightarrow A \times B \subset M \times N$ compact.

ix. Equivalence of any of the definitions of continuity on a whole space.

x. $f : M_1 \to M_2$ continuous, and $A \subset M_1$ compact implies $f(A)$ compact. Examples of continuous function in which the image of a closed set is not closed; example of a continuous function for which the image of a bounded set is not bounded.

xi. $f : M_1 \to M_2$ continuous, and $A \subset M_1$ compact implies f uniformly continuous on A. (Proof not required.) Examples in $\Re^n$ where this fails if compact is replaced by either closed or bounded.

xii. C compact in M, $A \subset C$, A closed $\Rightarrow A$ is compact.

xiii.

(d) Results in $\Re$ or $\Re^n$:

i. $s_n \to L$ and $t_n \to M \Rightarrow s_n + t_n \to L + M$. ($\epsilon, N$ proof)

ii. Other limit combination theorems (without proof)

iii. If x is any real number, its finite decimal approximations converge to x. (proof)

iv. $s_n \to L \Leftrightarrow \limsup\{s_n\}_{n=1}^{\infty} = \liminf\{s_n\}_{n=1}^{\infty} = L$ (without proof)

v. If $\limsup\{s_n\}_{n=1}^{\infty} = M < \infty$, then $\{s_n\}_{n=1}^{\infty}$ has a subsequence which converges to M. (similarly there is a subsequence which converges to $m = \liminf\{s_n\}_{n=1}^{\infty}$

vi. (MCT) A nondecreasing sequence that is bounded above converges to its least upper bound. (proof)

vii. $[a, b]$ is compact in $\Re$; $[a, b]^n$ is compact in $\Re^n$.

viii. Heine-Borel Theorem: $A \subset \Re^n \Rightarrow A$ compact.

ix. Max-min theorem: $f : \Re \to \Re$, f assumes its max and min on $[a, b]$.

3. (Pieces of) HW problems might be asked on the test. In particular: Ch 1 11, Ch 2: 1,2,3,6,7,9 (w/o proof).

4. Anything else we've covered that I think is easy.