Name:____________________________________

Real Variables, Math 5201

Midterm 2, Fall 2013

B. Peckham

The test has two parts. 100 points total.

Part I. Do all of the following problems. 50 points total.

1. (5 pts each, 20 pts total) Give examples of the following. If a metric space is not specified in the question, make sure you indicate the metric space(s) in which your examples live.

   (a) A set that is connected, but not path connected. Explain briefly why your set has both properties.

   (b) A continuous function $f : M \to N$ and a closed set $A \subset M$ for which $f(A)$ is not closed. State explicitly $M, N, f, A, f(A)$ for your example.

   (c) An open cover of $[0, 1) \subset \Re$ with no finite subcover. Explain briefly.

   (d) A function $f : \Re \to \Re$ which is in the function space $C^2(\Re, \Re)$, but not $C^3(\Re, \Re)$. Justify briefly.

2. (5 pts) Define what it means for $\mathcal{U}$ to be *an open cover of* $A \subset M$ and for $\mathcal{V}$ to be a *finite subcover* of $\mathcal{U}$ (which also covers A).

3. (5 pts) Explain how results from topology can be used to prove the following version of the Max-Min theorem from Calculus I. A formal proof is not required. If $f : [a, b] \to \Re$, and f is continuous, then there is a $c \in [a, b]$ with the property that $f(c) \geq f(x)$ for all $x \in [a, b]$.

4. (5 pts) Determine the largest possible Lebesgue number for the open cover $\mathcal{U} = \{(-2, 1/4), (0, 7/8), (1/2, 2)\}$ of $[0, 1]$. What point in $[0, 1]$ requires the Lebesgue number to be no bigger than your answer? No further justification necessary.

5. (5 pts) Give an example of a subset A of a metric space M which is bounded, but not totally bounded. State explicitly A, M (including the metric), a specific ball which contains A and an $r > 0$ for which there exists no finite collection of balls in M whose union contains A.

6. (5pts) Give an example of a subset A of a metric space M that is closed and bounded, but not compact. Explain briefly. Include explicit descriptions of A and M.

7. (5 pts) Explicitly list all types of connected sets in $\Re^1$. State which of these are also compact. No justfication necessary.

Part II. Do 5 of the following 6 proofs. 10 points each. 50 points total. You may do the remaining proof for 5 points extra credit. If you do not clearly mark 'EC' for the proof which is to be the extra credit proof, I will count number 6 as the extra credit.

1. Let M be a metric space. Prove that if $A \subset M$ is compact, then A is closed. You may use either the "sequential" or "open cover" definition of compactness.

2. Let M be a metric space. Prove that $A \subset M$ totally bounded implies that A is bounded.

3. Show that if M and N are metric spaces, $A \subset M$, A is covering compact, and $f : M \to N$ is continuous, then $f(A)$ is covering compact. Do not use sequential compactness.

4. Prove that if $A \subset M$ is path connected, then A is also connected. You may assume that $[a, b] \subset \Re$ is connected.

5. Let (a_n, b_n) be a sequence in $M \times N$, where M and N are metric spaces with respective metrics d_M and d_N. Assume this sequence converges in the "max" metric, defined as $d((a_1, b_1), (a_2, b_2)) = \max\{d_M(a_1, a_2), d_N(b_1, b_2)\}$. Prove that if $(a_n, b_n) \to (a, b)$ in $M \times N$, then $a_n \to a$ in M. Use an "ϵ-δ argument.

6. (EC unless another problem of 1-5 is marked as EC) Prove that if M is a metric space, $A \subset M$, and A is covering compact, then A is sequentially compact. You need not prove the "lemma" that if no subsequence of a given sequence (a_n) in A converges to the point $a \in A$, then there exists an $r > 0$ such that $a_n \in M_r a$ for only finitely many n.