Name:_____________________________________

Real Variables, Math 5201

Midterm 1, Fall 2013

B. Peckham

The test has two parts. 100 points total.

Part I. Do all of the following problems. 50 points total.

1. (25 pts) Give examples of the following. If your example is in any space other than $\mathbb{R}$ with the Euclidean metric, make sure you indicate the metric space(s) in which your examples live. **Justify all answers briefly.**

   (a) An infinite collection of closed sets F_i whose infinite union $A = \bigcup_{i=1}^{\infty} F_i$ is neither open nor closed. State explicitly what A is for your example.

   (b) A sequence $(a_n)_{n=1}^{\infty}$ in $\mathbb{R}$ which satisfies $\liminf_{n\to\infty} a_n = 2, \limsup_{n\to\infty} a_n = 100$ and no $a_n = 2$ or 100.

   (c) A metric space which is not complete. Explain briefly.

   (d) A metric space M, a submetricspace $N \subset M$, and a subset $U \subset N$ for which U is open in N, but not open in M. State explicitly M, N, U, and explain briefly why your example is correct.

   (e) A function f and a pair of sets A and B for which $f(A \bigcap B) \neq f(A) \bigcap f(B)$.

2. (10 pts) For each of the following sets A below, fill in the following table. Put Yes or No in the "?" columns according to whether the sets have the indicated properties. No justification required. E stands for Euclidean metric; d stands (only in this problem) for discrete metric; $\mathbb{R}$ is the real numbers; $\mathbb{Q}$ is the rational numbers; S is the metric subspace $[0, \infty)$ of $\mathbb{R}$ with the Euclidean metric.

Space	A	Open?	Closed?	Limits of A	Cluster pts of A
$(\mathbb{R}, \mathrm{E})$	$[0, 1)$				
$(\mathbb{R}, d)$	$[0, 1)$				
(S, E)	$[0, 1) \bigcap \mathbb{Q}$				

3. (5pts) Consider the function $f : \mathbb{R} \to \mathbb{R}$ where $f(x) = \chi_{[0,\infty)}(x)$. Define a sequence (a_n) in $\mathbb{R}$ which converges to some point l, but for which the corresponding sequence $f(a_n)$ does not converge to $f(l)$. State explicitly your sequence, l and $f(l)$.

4. (5pts) Consider the metric space $\mathbb{R}^2$ with the usual Euclidean metric. Let N be the submetric space consisting of the unit square ($\{(x, y) : 0 \leq x \leq 1, 0 \leq y \leq 1\}$). Sketch and label the ball $N_{0.5}(1, 1)$.

5. (5pts) Assume $f : \mathbb{R} \to \mathbb{R}$ and f satisfies $-5 \leq f'(x) \leq 10$. Must f be uniformly continuous on $\mathbb{R}$? Explain briefly why if yes; give a an example which satisfies the conditions and is not uniformly continuous if no.

6. EXTRA CREDIT (3pts): If we had not introduced rational cuts in this course, what would we have had to assume about the real numbers in order to proceed to theorems about the reals?

Part II. Do any 5 of the following 6 proofs. 10 points each. 50 points total. Mark clearly the proof you do NOT wish to count. Otherwise I will grade the first 5 proofs. No extra credit for the 6th proof. Proof NOT counted: 6 unless another number is here: ______.

1. Prove that a real sequence (x_n) that is increasing and bounded above converges. You may assume the least upper bound property of the reals.

2. Prove directly from first principles that if $a_n > 0$ and $\limsup_{n\to\infty} \frac{a_{n+1}}{a_n} = L < 1$, then $\sum_{n=1}^{\infty} a_n$ converges.

3. Show that if A_α is open in a metric space (M, d) for all $\alpha \in \mathcal{I}$ ($\mathcal{I}$ is an arbitrary index set), then $A = \bigcup_{\alpha\in\mathcal{I}} A_\alpha$ is open in (M, d).

4. Prove directly that the rationals are countable.

5. Let $f : (M, d_1) \to (N, d_2)$. Let $m_0 \in M$. Prove that if f satisfies the "open set" definition for being continuous on M, then f satisfies the ϵ-δ definition of being continuous at any point $m_0 \in M$. Include both definitions.

6. Prove that a Cauchy sequence in a metric space is bounded.