MIDTERM 1 (Wednesday, October 7 2015, 3-4:30 in SCC 130)
TOPIC LIST
(as of October 5, 2015)
Real Variables, Math 5201
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1. Definitions and Notation
   (a) Operations on sets: $A \cup B, A \cap B, A \backslash B, A^c$
   (b) $A \times B$
   (c) Function, domain, codomain=target, image=range
   (d) $f(A), f^{-1}(A)$ (Image and inverse image of A)
   (e) $\mathbb{N} \subset \mathbb{Z} \subset \mathbb{Q} \subset \mathbb{R}$
   (f) Definition of $\mathbb{R}$ as the collection of all "cuts" of the rationals.
   (g) For a set $A \subset \mathbb{R}$, upper bound, lower bound, Least upper bound (lub), greatest lower bound (glb)
   (h) For $\mathbb{R}$: least upper bound property, greatest upper bound property
   (i) Metric Space: M or (M, d), subspace, (Cartesian) product space
   (j) Complete metric space (every Cauchy sequence converges)
   (k) Topological Space, subspace, (Cartesian) product space
   (l) For a set $A \subset M$: bounded, unbounded
   (m) χ_A (or 1_A, the indicator function of A)
   (n) Sequence in M, subsequence, bounded, unbounded; monotone in $\mathbb{R}$
   (o) ϵ, δ definition for the limit of a function $f : M \to N$, as $x \to a$, including one-sided limits when $M = \mathbb{R}$.
   (p) ϵ, N definition for a sequence to i) converge to L, ii) be Cauchy. (in $\mathbb{R}$ or M)
   (q) $\limsup a_n$ and $\liminf a_n$ of a real sequence (a_n)
   (r) M, N definition for a sequence in $\mathbb{R}$ to diverge to $+\infty$ (or $-\infty$).
   (s) Definition for a real series to converge.
   (t) Cardinality: $A \sim B$ if there is a bijection between A and B. (A is "equinumerous" with B, or A has the same cardinality as B)
   (u) A is countably infinite ($A \sim \mathbb{N}$)
   (v) Topology (in metric spaces): open, closed, limits of A, accumulation (cluster) points of A, $\overline{A} = cl(A) = cl_M(A) =$ the closure of A (in M), $A^\circ = int(A) =$ the interior of A, $\partial A =$ the boundary of A, open ball $= M_r x = B(x, r) = B_r(x)$
   (w) homeomorphism
   (x) Continuity of $f : M_1 \to M_2$ at a:
      i. ϵ, δ
      ii. $x_n \to a \Rightarrow f(x_n) \to f(a)$
   (y) Continuity of $f : M_1 \to M_2$ on M_1:
      i. Continuous at each $a \in M_1$ using any of the above notions of continuous at a point.
      ii. Inverse image of any open set is open
      iii. Inverse image of any closed set is closed
   (z) Uniform continuity of $f : M_1 \to M_2$ on M_1. (ϵ-δ)

2. Examples, results and constructions to know:

(a) Definition of a "candidate" for the least upper bound of a subset $S \subset \mathbb{R}$ using the ideas and notation of cuts.

(b) "Candidate for the limit of a Cauchy (bounded) sequence in $\mathbb{R}$.

(c) Relationships between sets:

   i. DeMorgan's laws: $(A \cup B)^c = A^c \cap B^c$; $(A \cap B)^c = A^c \cup B^c$

   ii. $f^{-1}(A \cup B) = f^{-1}(A) \cup f^{-1}(B)$

   iii. $f^{-1}(A \cap B) = f^{-1}(A) \cap f^{-1}(B)$

   iv. $f(A \cup B) = f(A) \cup f(B)$

   v. $f(A \cap B) \subseteq f(A) \cap f(B)$. Know an example showing strict inclusion.

   vi. $f(A) \backslash f(B) \subseteq f(A \backslash B)$. Know an example showing strict inclusion.

(d) Series convergence tests: geometric, basic comparison, limit comparison, integral test, ratio test (with limsup/liminf), root test (with limsup/liminf)

(e) Specific metrics for $\mathbb{R}^n$: Euclidean, max, sum

(f) The discrete metric (d(x,y) = 0 or 1) for any metric space

(g) Examples: A collection of closed sets whose union is not closed; a collection of open sets whose intersection is not open.

3. Proofs to know:

   (a) Sequence and series proofs:

      i. Proof that the limit of a specific function or sequence exists.

      ii. A sequence that converges is Cauchy (in $\mathbb{R}$ or M)

      iii. A Cauchy sequence is bounded (in $\mathbb{R}$ or M)

      iv. (MCT) A nondecreasing real sequence that is bounded above converges to its least upper bound.

      v. Basic comparison test for convergence/divergence of a series of positive terms. (Show partial sums are bounded.)

      vi. Proof that a specific p-series converges/diverges using the proof techniques of the integral test (not just quoting the integral test). (Show partial sums are bounded.)

      vii. Proof of the ratio test for convergence of a series with positive terms in the case that $\limsup a_{n+1}/a_n = L < 1$. (Show partial sums are bounded.)

   (b) Cardinality equivalences and inequivalences.

      i. $\mathbb{N} \sim 2\mathbb{N}$. (The natural numbers are equivalent to the even natural numbers.) Explicit construction.

      ii. $\mathbb{Q}$ is countable. More generally, countable unions of countable sets are countable.

      iii. If A and B are countable, so is $A \times B$.

      iv. The unit interval $[0, 1]$ is not countable. Assume $[0, 1]$ is the set of infinite decimal expansions which do not end in an infinite string of 9's. (Proof by contradiction.)

      v. $\sqrt{2}$ is irrational

   (c) Metric Space results:

      i. Properties of open and closed sets, for example A_α open $\Rightarrow \cup A_\alpha$, $A_1 \cap A_2$ both are open, $\overline{A}$ is closed, $M_r x$ is open, $\overline{A} = limits(A)$.

      ii. Show A open in $M \Leftrightarrow A^c$ closed in M.

      iii. Uniqueness of limit of a convergent sequence (proof) (ϵ, N proof)

      iv. Any direction of equivalence of ϵ-δ, sequence, and open set definitions of continuity on a whole space.

4. (Pieces of) HW problems might be asked on the test. In particular: Ch 1: 2a, 11; Ch 2: 3,6b,7

5. Anything else we've covered that I think is easy.