Name:____________________________________

Real Variables, Math 5201

Midterm 1, Fall 2014

B. Peckham

The test has two parts. 100 points total.

Part I. Do all of the following problems. 50 points total.

1. (20 pts) Give examples of the following. If your example is in any space other than $\mathbb{R}$ with the Euclidean metric, make sure you indicate the metric space(s) in which your examples live. **Justify all answers briefly.**

   (a) An infinite collection of open sets G_i whose infinite intersection $A = \bigcap_{i=1}^{\infty} G_i$ is not open. State explicitly what A is for your example.

   (b) Two different metrics d_1 and d_2, a space M and a sequence $\{a_n\}$ in M which converges in the metric space (M, d_1), but does not converge in the metric space (M, d_2). Explain briefly.

   (c) A metric space M, a submetricspace $N \subset M$, and a subset $U \subset N$ for which U is closed in N, but not closed in M. State explicitly M, N, U, and explain briefly why your example is correct.

   (d) Consider the following sequence:

   $$a_n = \begin{cases} \frac{1}{n^2} & n \text{ odd} \\ \frac{1}{n^3} & n \text{ even} \end{cases}$$

   Compute

   $$\limsup_{n\to\infty} \frac{a_{n+1}}{a_n}.$$

   Does

   $$\sum_{n=1}^{\infty} a_n$$

   converge? Justify briefly.

2. (10pts) For each of the following sets A below, fill in the following table. Put Yes or No in the "?" columns according to whether the sets have the indicated properties. No justification required. E stands for Euclidean metric; d stands (only in this problem) for discrete metric; $\mathbb{R}$ is the real numbers; S is the metric subspace $(0,1] \cup [2,3)$ of $\mathbb{R}$ with the Euclidean metric.

Space	A	$\overline{A}$ (closure)	A° (interior)	Open?	Closed?
($\mathbb{R}$, E)	$[0,1)$				
$(\mathbb{R}, d)$	$[0,1)$				
(S, E)	$\{\frac{1}{n}\}_{n=1}^{\infty}$				

3. (5pts) Give an example of a function $f : M \to N$ and an open set $U \subset N$ such that $f^{-1}(U)$ is not open in M.

4. (5 pts) Explain why $\sqrt{2}$ is not rational. (Formal proof not necessary, just a convincing argument.)

5. (5 pts) Explain what we can conclude from the existence of a surjection from a set A onto a set B about the cardinalities of A and B. Explain briefly.

6. (5 pts) Give an explict bijection between the natural numbers and the even integers. In order to receive full credit, your bijection must either be described by explicit formula, or by algorithm, such as a recursive formula.

Part II. Do any 5 of the following 6 proofs. 10 points each. 50 points total. Mark clearly the proof you do NOT wish to count. Otherwise I will grade the first 5 proofs. The designated problem can be counted for up to 5 extra credit points.

Proof for extra credit: ______.

1. Prove that a real sequence (x_n) that is increasing and bounded above converges. You may assume the reals have the least upper bound property.

2. Prove directly from first principles that if $a_n > 0$ and $\limsup_{n\to\infty} \frac{a_{n+1}}{a_n} = L < 1$, then $\sum_{n=1}^{\infty} a_n$ converges. You may use the monotonic convergence theorem from problem 1 even if you don't do problem 1.

3. Show that if A and B are open in a metric space (M, d), then $A \bigcap B$ is open in (M, d). Prove directly from the definition of open sets (without using DeMorgan's laws and closed set properties).

4. Do one of the following two proofs, but not both.

   (a) Prove directly from the definitions of open and closed sets that if $A \subset M$ is closed in M, then its complement, A^c, is open in M.

   (b) Prove that $M_r x$ is open. Include a definition of $M_r x$.

5. Let $f : (M, d_1) \to (N, d_2)$. Let $m_0 \in M$. Prove that if f satisfies the ϵ-δ definition of being continuous at m_0, then f will also satisfy the "sequence" definition of being continuous at m_0. Include both definitions.

6. (5 pts extra credit unless another problem is designated extra credit) Prove that a Cauchy sequence in a metric space (M, d) is bounded.