

TOPIC LIST

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Real Variables, Math 5201

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1. Definitions and Notation

- (a) Operations on sets: $A \cup B, A \cap B, A \setminus B, A^c$
- (b) $A \times B$
- (c) Function, domain, codomain=target, image=range
- (d) $f(A), f^{-1}(A)$ (Image and inverse image of A)
- (e) $\mathbb{N} \subset \mathbb{Z} \subset \mathbb{Q} \subset \mathbb{R}$
- (f) Definition of \mathbb{R} as the collection of all “cuts” of the rationals.
- (g) For a set $A \subset \mathbb{R}$, upper bound, lower bound, Least upper bound (lub), greatest lower bound (glb)
- (h) For \mathbb{R} : least upper bound property, greatest upper bound property
- (i) Metric Space: M or (M, d) , subspace, (Cartesian) product space
- (j) Complete metric space (every Cauchy sequence converges)
- (k) Topological Space, subspace, (Cartesian) product space
- (l) For a set $A \subset M$: bounded, unbounded
- (m) χ_A (or 1_A , the indicator function of A)
- (n) Sequence in M , subsequence, bounded, unbounded; monotone in \mathbb{R}
- (o) ϵ, δ definition for the limit of a function $f : M \rightarrow N$, as $x \rightarrow a$, including one-sided limits when $M = \mathbb{R}$.
- (p) ϵ, N definition for a sequence to i) converge to L , ii) be Cauchy. (in \mathbb{R} or M)
- (q) $\limsup a_n$ and $\liminf a_n$ of a real sequence (a_n)
- (r) M, N definition for a sequence in \mathbb{R} to diverge to $+\infty$ (or $-\infty$).
- (s) Definition for a real series to converge.
- (t) Cardinality: $A \sim B$ if there is a bijection between A and B . (A is “equinumerous” with B , or A has the same cardinality as B)
- (u) A is countably infinite ($A \sim \mathbb{N}$)
- (v) Topology (in metric spaces): open, closed, limits of A , accumulation (cluster) points of A , $\bar{A} = cl(A) = cl_M(A)$ = the closure of A (in M), $A^\circ = int(A)$ = the interior of A , ∂A = the boundary of A , open ball = $M_r x = B(x, r) = B_r(x)$
- (w) homeomorphism
- (x) Continuity of $f : M_1 \rightarrow M_2$ at a :
 - i. ϵ, δ
 - ii. $x_n \rightarrow a \Rightarrow f(x_n) \rightarrow f(a)$
- (y) Continuity of $f : M_1 \rightarrow M_2$ on M_1 :
 - i. Continuous at each $a \in M_1$ using any of the above notions of continuous at a point.
 - ii. Inverse image of any open set is open
 - iii. Inverse image of any closed set is closed
- (z) Uniform continuity of $f : M_1 \rightarrow M_2$ on M_1 . (ϵ - δ)

2. Examples, results and constructions to know:

- (a) Definition of a “candidate” for the least upper bound of a subset $S \subset \mathbb{R}$ using the ideas and notation of cuts.
 - (b) “Candidate for the limit of a Cauchy (bounded) sequence in \mathbb{R} .”
 - (c) Relationships between sets:
 - i. DeMorgan’s laws: $(A \cup B)^c = A^c \cap B^c$; $(A \cap B)^c = A^c \cup B^c$
 - ii. $f^{-1}(A \cup B) = f^{-1}(A) \cup f^{-1}(B)$
 - iii. $f^{-1}(A \cap B) = f^{-1}(A) \cap f^{-1}(B)$
 - iv. $f(A \cup B) = f(A) \cup f(B)$
 - v. $f(A \cap B) \subseteq f(A) \cap f(B)$. Know an example showing strict inclusion.
 - vi. $f(A) \setminus f(B) \subseteq f(A \setminus B)$. Know an example showing strict inclusion.
 - (d) Series convergence tests: geometric, basic comparison, limit comparison, integral test, ratio test (with limsup/liminf), root test (with limsup/liminf)
 - (e) Specific metrics for \mathbb{R}^n : Euclidean, max, sum
 - (f) The discrete metric ($d(x,y) = 0$ or 1) for any metric space
 - (g) Examples: A collection of closed sets whose union is not closed; a collection of open sets whose intersection is not open.
3. Proofs to know:
- (a) Sequence and series proofs:
 - i. Proof that the limit of a specific function or sequence exists.
 - ii. A sequence that converges is Cauchy (in \mathbb{R} or M)
 - iii. A Cauchy sequence is bounded (in \mathbb{R} or M)
 - iv. (MCT) A nondecreasing real sequence that is bounded above converges to its least upper bound.
 - v. Basic comparison test for convergence/divergence of a series of positive terms. (Show partial sums are bounded.)
 - vi. Proof that a specific p-series converges/diverges using the proof techniques of the integral test (not just quoting the integral test). (Show partial sums are bounded.)
 - vii. Proof of the ratio test for convergence of a series with positive terms in the case that $\limsup a_{n+1}/a_n = L < 1$. (Show partial sums are bounded.)
 - (b) Cardinality equivalences and inequivalences.
 - i. $\mathbb{N} \sim 2\mathbb{N}$. (The natural numbers are equivalent to the even natural numbers.) Explicit construction.
 - ii. \mathbb{Q} is countable. More generally, countable unions of countable sets are countable.
 - iii. If A and B are countable, so is $A \times B$.
 - iv. The unit interval $[0, 1]$ is not countable. Assume $[0, 1]$ is the set of infinite decimal expansions which do not end in an infinite string of 9’s. (Proof by contradiction.)
 - v. $\sqrt{2}$ is irrational
 - (c) Metric Space results:
 - i. Properties of open and closed sets, for example A_α open $\Rightarrow \cup A_\alpha$, $A_1 \cap A_2$ both are open, \overline{A} is closed, $M_r x$ is open, $\overline{A} = \lim(A)$.
 - ii. Show A open in $M \Leftrightarrow A^c$ closed in M .
 - iii. Uniqueness of limit of a convergent sequence (proof) (ϵ, N proof)
 - iv. Any direction of equivalence of ϵ - δ , sequence, and open set definitions of continuity on a whole space.
4. (Pieces of) HW problems might be asked on the test. In particular: Ch 1: 2a, 11; Ch 2: 3,6b,7
5. Anything else we’ve covered that I think is easy.