MIDTERM 2 TOPIC LIST: For midterm Wed. Nov. 16, 2016
(as of November 17, 2016)
Real Variables, Math 5201
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1. Definitions

   (a) Basic metric space definitions, including convergence of a sequence, open, closed, interior (interior point or smallest open set definition), closure (limit point or smallest closed set definition), boundary
   (b) Sub(metric)space topology; inheritance principle for open and closed sets
   (c) Four equivalent definitions for $f : M \to N$ to be continuous (ϵ-δ, sequential, inverse of open is open, inverse of closed is closed)
   (d) homeomorphism (1-1, onto, continuous, continuous inverse)
   (e) Sequentially compact
   (f) Open cover of a subset; subcover; finite subcover; LebeAresgue number of a cover of a set.
   (g) Covering compact
   (h) Connected space, connected subspace (subset), disconnected subset/subspace
   (i) Path connected
   (j) Component, path component
   (k) Diameter of a set: diam(A)
   (l) Bounded, totally bounded
   (m) A a dense subset of B (three equivalent definitions: closure, sequential, point within ϵ)
   (n) In $\Re$: (TEST 1 or TEST 3)
      i. f is differentiable at x (definition of derivative)
      ii. convergence/divergence of a sequence
      iii. Infinite series, convergence/divergence of a series
      iv. Conditional convergence vs absolute convergence (using partial sums)
      v. Rearrangement of a series

2. Proofs to know:

   (a) $A \subset M$, A open implies A^c closed; A closed implies A^c open.
   (b) $A \subset M$, A totally bounded implies A bounded.
   (c) Sequentially compact sets are closed and bounded.
   (d) Covering compact sets are closed, totally bounded, and bounded.
   (e) Closed subsets of (sequentially/covering) compact sets are (sequentially/covering) compact. Know the proof in both settings.
   (f) Prove that continuous maps take compact sets to compact sets. Know proof using either sequential compactness or covering compactness.
   (g) Prove that continuous maps take connected sets to connected sets.
   (h) Prove that $f : M \to N$ a homeomorphism implies that $f : M \backslash A \to N \backslash f(A)$ is a homeomorphism.(Sequential def of continuity is probably the easiest to use.)
   (i) Prove path connected implies connected.
   (j) Product space theorem: Prove any of 4 convergence notions implies any other: convergence in each component, convergence in product space using Euclidean, sum, or max metric
   (k) Prove that any of the three definitions of A dense in B in a metric space M is equivalent to any other.
   (l) Prove that if no subsequence of a given sequence (a_n) converges to $m \in M$, then there exists an $r > 0$ such that $a_n \in M_r m$ for only finitely many n.
   (m) Prove that if $A \subset M$ is open cover compact, then it is sequentially compact.

(n) Calculus results:

i. (TEST 1) Geometric series: Prove that $\sum_{n=0}^{\infty} x^n = \frac{1}{1-x}$ for $|x| < 1$.

ii. (TEST 1) p-series.

A. Prove that $\sum_{n=1}^{\infty} \frac{1}{n}$ (show partial sums are not bounded by using the integral test, or by grouping terms).

B. Prove that $\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges using the integral test.

C. know that $\sum_{n=1}^{\infty} \frac{1}{n^p}$ converges iff $p > 1$.

iii. (TEST 1) Prove the the ratio test: $\limsup_{n\to\infty} \frac{|a_{n+1}|}{|a_n|} < 1$ implies absolute convergence.

iv. (TEST 1) Prove the basic comparison test in the special case that all terms in both series are nonnegative. However, treat the case that the comparison is valid only from n beyond some N.

v. Prove the Calculus I Max-min theorem for continuous $f : [a, b] \to \Re$ (OK to quote results for compact and connected sets)

vi. Prove the Calculus I Intermediate value theorem for continuous $f : [a, b] \to \Re$ (OK to quote results for compact and connected sets)

3. Miscellaneous results and techniques to know:

(a) (TEST 3) Characterization of $\sum b_n, \sum c_n$ when $\sum a_n$ is absolutely convergent, conditionally convergent, or divergent. ($\{b_n\}$ is the subsequence of nonnegative terms in $\{a_n\}$, $\{c_n\}$ is the subsequence of negative terms in $\{a_n\}$.

(b) (TEST 1) Decide whether given series converge or diverge.

(c) (TEST 1) Series examples: conditionally convergent; alternating, terms converging to zero but the series diverges, $\limsup a_{n+1}/a_n > 1$ but the series converges.

(d) Characterization in $\Re^1$ of all connected sets, compact connected sets, open connected sets.

(e) Example of a set that is connected but not path connected

(f) Example of an open cover of a (noncompact) set with no finite subcover; a cover with no positive Lebesgue number; a set that is bounded but not totally bounded; a set that is closed and bounded but not compact.

(g) Example of a (noncompact) set which has a sequence with no convergent subsequence

(h) Identify (non)homeomorphic sets

(i) Examples of f continuous, A closed, $f(A)$ not closed; f continuous, A bounded, $f(A)$ not bounded.

(j) Examples of a discontinuous function (or for any given discontinuous function) $F : M \to N$ and 1) an $\epsilon > 0$ and m in M for which there "exists no delta", 2) a sequence (a_n) in M that converges to l for which $f(a_n)$ does not converge to $f(l)$, 3) an open set in N for which the preimage is not open in M, 4) a closed set in N for which the preimage is not closed in M.

(k) (TEST 3) Examples of functions that are C^0 but not C^1, C^k but not C^{k+1}, differentiable but not C^1, C^∞, but not analytic, discontinuous everywhere, continuous on the irrationals but not the rationals.

4. Easy HW problems. Especially: Ch 2: 6,8,22,47, 48, 54, 55

5. Anything else we've covered that I think is easy.