MIDTERM 2 TOPIC LIST (as of December 3, 2009)
Dynamical Systems, Math 5260
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1. Differential Equations
   (a) Know advantages/disadvantages of phase space analysis vs finding analytical solutions.
   (b) For 1D autonomous differential equations ($\dot{x} = f(x)$), sketch a phase line. Given any one of the following, sketch the other two.
      i. phase line
      ii. graph of f
      iii. sketch of several solutions (in the "x-t" plane)
   (c) Phase planes for 2D autonomous differential equations
      i. Sketch phase planes for: decoupled, one of the 3 "canonical" linear forms, or given in polar coordinates.
      ii. Given a phase plane with several orbits, identify equilibrium points, characteristics of eigenvalues, and any real eigenspaces, stable and unstable manifolds of saddles. If a formula for the differential equation is given, determine the direction of the arrows on the oribits. Describe long term behavior of orbits and how this behavior depends on initial conditions.
      iii. Nullclines. Given a formula for a 2D differential equation, find and sketch the nullclines. Determine the direction of the arrows along the nullclines. Given a phase plane with several orbits (but no equations) sketch the nullclines.
      iv. Determine whether any of the coordinate axes in a differential equation are invariant.
      v. Sketch solution curves ($x_1(t)$ and $x_2(t)$) consistent with phase portraits. Sketch phase orbits from solution curves.
   (d) Definitions and Terminology: Solution, maximal interval of existence, flow, orbit, slope field (1D), vector field (1D,2D,or 3D), direction field (1D, 2D, or 3D)
   (e) Be able to locate equilibrium points (fixed points) for 1D, 2D and 3D (autonomous) differential equations and assign whatever of the following labels that apply: hyperbolic, nonhyperbolic, attracting, repelling, saddle, stable, asymptotically stable, unstable.
   (f) Find the linear differential equation that approximates a nonlinear differential equation near an equilibrium point (1D, 2D and 3D). Know when the phase portrait of the linearized differential equation is "equivalent" to the full nonlinear phase portrait near an equilibrium point.
   (g) Given a 1D or 2D differential equation and a change of variables, find the differential equation in the new variables.
   (h) Show the eigenvalues of a 2 × 2 matrix $A$ satisfy: $\lambda^2 - tr(A)\lambda + det(A) = 0$. Know and understand the $det(A)$ vs. $tr(A)$ diagram.
   (i) Existence–uniqueness theorem (same for 1D,2D and 3D): $f \in C^0$ implies existence of a solution to an initial value problem: $\dot{x} = f(x), x(0) = x_0$; $C^1$ implies uniqueness as well. Implies solutions cannot cross in the phase space.
   (j) Convert a second order system of differential equations into a system of first order differential equations.
   (k) Bifurcations
      i. Given a family (1 or 2 parameters) of 1D differential equations, locate – graphically or analytically – the equilibria and (local) bifurcations sets in the phase × parameter space and their projections to the parameter space.
      ii. Recognize a saddle-node, transcritical, and pitchfork bifurcations in a one-parameter – one-phase-dimension bifurcation diagram.
      iii. Write down an example of a saddle-node, transcritical, or pitchfork bifurcation in 1D or 2D. Write down an example of a Hopf bifurcation in 2D (easiest in polar coordinates).

iv. Locate bifurcations (saddle-node, transcritical, pitchfork or Hopf) in 2D or 3D families of differential equations. First 3: equilibrium point with eigenvalue of zero; Hopf: equilibrium point with pure imaginary eigenvalues.

(l) Compute explicit solutions for 1D linear ($\dot{x} = ax$) and 2D diagonal linear ($\dot{x} = Ax$, $A$ a diagonal matrix).

2. One Dimensional Maps: $x_{n+1} = f(x_n)$ (real) and $z_{n+1} = f(z_n)$ (complex).

   (a) Definitions to know:
      i. Fixed point, periodic point, orbit, cycle, period, prime or least period
      ii. Attracting, superattracting, repelling, (linearly) neutral periodic point (orbit, cycle); use of chain rule in determining these adjectives
      iii. Phase portrait (for orientation preserving real homeomorphisms only)
      iv. Graphical analysis (real)
      v. Bifurcation (change in qualitative description of solutions, including a change in the number or stability of fixed/periodic orbits)
      vi. The sequence space $\Sigma$, the "usual" metric on $\Sigma$, the shift map $\sigma$.
      vii. Three properties of a chaotic system and all terminology used in the def. of the properties.
      viii. Homeomorphism, topological conjugacy, topological semiconjugacy
      ix. $z$ plane: Filled Julia Set, Julia Set, dichotomy or trichotomy
      x. $C$ plane: Mandelbrot set, dichotomy

   (b) Results to know:
      i. Relationship between the shift map on the symbol sequence space and the quadratic map (for $x^2 + c$ with $c < -2$ or $z^2 + c$ with $|c| > 2$) on the invariant Cantor set.
      ii. Escape algorithm for $z^2 + c$.

   (c) Techniques to know:
      i. Locating fixed and periodic pts/orbits analytically (for individual maps and for families of maps)
      ii. Locating period-$n$ pts/orbits of $f$ from graphs of $f$ and $f^n$. (real)
      iii. Interpreting orbit diagrams (identifying, for example, parameter values corresponding to maps with attracting orbits of a certain period, or saddle-node bifurcations or period-doubling bifurcations) (real)
      iv. Determining stability of periodic orbits either analytically (real or complex)or from graphs (real).
      v. Constructing a graph of $f$ so that $f$ has, for example, a periodic orbit of a certain period and certain derivative (of $f^n$). (real)
      vi. Constructing the graph of iterates of $f$ given the graph of $f$. (real)
      vii. The construction of the invariant Cantor set $\Lambda$ for $x^2 + c$ with $c < -2$ (discard middle intervals) or $z^2 + c$ with $|c| > 2$ (figure eights).
      viii. Given a map and a change of variables, find the equation for the map in the new variables.
      ix. Recognizing a saddle-node and/or period doubling bifurcation from a sequence of graphs of a family of maps as a parameter changes. (real)
      x. Locating parameter values for which a corresponding family of maps (either real or complex) has any one of the following: a superattracting orbit (derivative 0), a saddle-node bifurcation (derivative 1), a period-doubling bifurcation (derivative -1), neutral periodic points (magnitude of the derivative = 1)
      xi. Locating various period-$q$ bulbs; locating $p/q$ bulbs. Recognizing bulb/decoration from Mandelbrot set corresponding to a given Julia set.

3. Anything else we've covered that I think is easy.