Real Analysis

Math 8201

Final Problem Set:

Due: Mon. May 11, 2009

DRAFT as of April 13, 2009

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Directions: Do all problems. You may use your text book and any other inanimate references. Indicate precisely any results you use from any references as well as the references themselves. Have fun.

1. A bad way to define inner and outer measure. Define λ^{**} and λ_{**} by

$$\lambda^{**}(A) = \inf\{\lambda(K) : A \subset K, K \text{ compact}\}$$

$$\lambda_{**}(A) = \sup\{\lambda(G) : G \subset A, G \text{ open}\}$$

   Explain why these definitions are not "as good" as Jones' definitions of λ^* and λ_*. An example of sets which the above definitions handle poorly would be appropriate in the discussion.

2. Consider the following alternative definition of Lebesgue measure.

   (a) Let $I = \{x \in \Re^n : a_j \le x_j \le b_j, j = 1, 2, ..., n\}$ be an *interval* in $\Re^n$. Define $\mu(I) = \prod_{j=1}^{n}(b_j - a_j)$.

   (b) A shorter alternative way to define Lebesgue measure. For any $E \subset \Re^n$, define $\mu_e(E) = \inf\{\sum_{k=1}^{\infty} \mu(I_j)\}$, where $E \subset \bigcup_{k=1}^{\infty} I_k$, each I_k an interval.

   (c) Say $E \subset \Re^n$ is μ-measurable if, given $\epsilon > 0$, there exists an open set G such that $E \subset G$, and $\mu_e(G \sim E) < \epsilon$.

   (d) If $E \subset \Re^n$ is μ-measurable, define $\mu(E) = \mu_e(E)$

   Show the following:

   (a) $\mu_e = \lambda^*$.

   (b) A set $E \subset \Re^n$ is μ-measurable if and only if E is Lebesgue measurable.

   (c) If $E \subset \Re^n$ is μ-measurable (and/or Lebesgue measurable), then $\mu(E) = \lambda(E)$.

3. Convolutions and approximation of functions in L^1 by nicer (continuous or C^1) functions. Let $f(x) = \chi_{[0,1)}(x)$. Compute explicit formulas for $f * \phi_a(x)$ for the following two choices of ϕ. You may assume $a < 1$. Determine how smooth the $f * \phi_a$ functions are (eg. are they C^k for some k?). Then show that in each case $f * \phi_a$ converges to f "in L^1."

   (a) $\phi(x) = \frac{1}{2}\chi_{[-1,1]}$

   (b) $\phi(x) = \begin{cases} x+1 & \text{for } -1 < x < 0, \\ -x+1 & \text{for } 0 < x \le 1, \\ 0 & \text{otherwise.} \end{cases}$

4. An alternative way to show the Cantor Lebesgue function is continuous. Define the sequence of functions $\{f_k\}$ recursively as follows:

   - $f_0(x) = x$ (for $x \in [0, 1]$.
   - f_1 is piecewise linear, connecting points $(0, 0)$ to $(\frac{1}{3}, \frac{1}{2})$ to $(\frac{2}{3}, \frac{1}{2})$ to $(1, 1)$.
   - f_{k+1} is constructed from f_k by making it piecewise linear, equal to f_k wherever f_k is constant, nondecreasing, and constant on middle thirds of intervals where f_k was increasing. Make the value of f_{k+1} equal to the average value of the endpoints of the interval from f_k.

   Show

   (a) $f_k(x)$ is a Cauchy sequence for all $x \in [0, 1]$. Call the limit function f. (This is the Cantor-Lebesgue function.)

   (b) f_k converges uniformly to f.

   (c) f is continuous. (This is essentially the proof that a sequence of continuous functions that converges uniformly has a continuous limit.)

5. Ch. 8: 1,3 (OK to assume the result of 2) ,4, 11 (for n=1 and 2 only), 12 Hint for 1: For one order of integration, substitute $u = xy$. Then show that the integral is the square of $\int_0^\infty e^{-t^2} dt$.