Directions: Do all problems. You may use your text book and any other inanimate references. Indicate precisely any results you use from any references as well as the references themselves. Have fun.

1. A bad way to define inner and outer measure. Define $\lambda^{**}$ and $\lambda^{* *}$ by

$$
\lambda^{**}(A) = \inf \{ \lambda(K) : A \subset K, K \text{ compact} \}
$$

$$
\lambda^{* *}(A) = \sup \{ \lambda(G) : G \subset A, G \text{ open} \}
$$

Give reasons why these definitions are not “as good” as Jones’ definitions of $\lambda^*$ and $\lambda_*$. Examples of sets which the above definitions handle poorly would be appropriate in the discussion.

2. Consider the following alternative definition of Lebesgue measure.

(a) Let $I = \{ x \in \mathbb{R}^n : a_j \leq x_j \leq b_j, j = 1, 2, ..., n \}$ be an interval in $\mathbb{R}^n$. Define $\mu(I) = \prod_{j=1}^{n} (b_j - a_j)$.

(b) A shorter alternative way to define Lebesgue measure. For any $E \subset \mathbb{R}^n$, define $\mu_e(E) = \inf \{ \sum_{k=1}^{\infty} \mu(I_j) \}$, where $E \subset \bigcup_{k=1}^{\infty} I_k$, each $I_k$ an interval.

(c) Say $E \subset \mathbb{R}^n$ is $\mu$-measurable if, given $\epsilon > 0$, there exists an open set $G$ such that $E \subset G$, and $\mu_e(G \sim E) < \epsilon$.

(d) If $E \subset \mathbb{R}^n$ is $\mu$-measurable, define $\mu(E) = \mu_e(E)$

Show the following:

(a) $\mu_e = \lambda^*$.

(b) A set $E \subset \mathbb{R}^n$ is $\mu$-measurable if and only if $E$ is Lebesgue measurable.

(c) If $E \subset \mathbb{R}^n$ is $\mu$-measurable (and/or Lebesgue measurable), then $\mu(E) = \lambda(E)$.

3. Convolutions and approximation of functions in $L^1$ by nicer (continuous or $C^1$) functions. Let $f(x) = \chi_{[0,1]}(x)$. Compute explicit formulas for $f * \phi_a(x)$ for the following two choices of $\phi$. You may assume $a < \frac{1}{2}$.

Sketch the graph of the convolution. You may use software such as Mathematica if you wish. Determine how smooth the $f * \phi_a$ functions are (e.g. are they $C^k$ for some $k$?). Then show directly (rather than quoting a general theorem) that in each case $f * \phi_a$ converges to $f$ “in $L^1$.”

(a) $\phi(x) = \frac{1}{2} \chi_{[-1,1]}$
(b) \( \phi(x) = \begin{cases} 
  x + 1 & \text{for } -1 < x < 0, \\
  -x + 1 & \text{for } 0 < x \leq 1, \\
  0 & \text{otherwise.} 
\end{cases} \)

4. An alternative way to show the Cantor Lebesgue function is continuous. Define the sequence of functions \( \{f_k\} \) recursively as follows:

- \( f_0(x) = x \) (for \( x \in [0, 1] \)).
- \( f_1 \) is piecewise linear, connecting points \((0, 0)\) to \((\frac{1}{3}, \frac{1}{2})\) to \((\frac{2}{3}, \frac{1}{2})\) to \((1, 1)\).
- \( f_{k+1} \) is constructed from \( f_k \) by making it piecewise linear, equal to \( f_k \) wherever \( f_k \) is constant, nondecreasing, and constant on middle thirds of intervals where \( f_k \) was increasing. Make the value of \( f_{k+1} \) equal to the average value of the endpoints of the interval from \( f_k \).

Show

(a) \( f_k(x) \) is a Cauchy sequence for all \( x \in [0, 1] \). Call the limit function \( f \). (This is the Cantor-Lebesgue function.)

(b) \( f_k \) converges uniformly to \( f \).

(c) \( f \) is continuous. (This is essentially the proof that a sequence of continuous functions that converges uniformly has a continuous limit.)

5. Ch. 8: 1,3 (OK to assume the result of 2) , 11 (for n=1 only), 12. Extra Credit: 4, 11 for n=2. Hint for 1: For one order of integration, substitute \( u = xy \). Then show that the integral is the square of \( \int_0^\infty e^{-t^2} \, dt \).