Reversing the Escape Algorithm for the Plane

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Contents

Abstract

The objective of this research is to develop a method of finding a function which produces a given escape diagram. An escape diagram is a graphical representation of the set of inputs in the domain of a function which remain bounded under iteration of the function. This research considers functions which have domains that are subsets of the complex numbers, although the functions are not necessarily complex analytic. The findings of this research include two methods: the fixed point method, and the conjugate method. The fixed point method provides a set of conditions which are contingent on the boundary of the given escape diagram consisting of repulsive fixed points (the boundary must be given as two functions defined on the interval (y_1, y_2)). If a function satisfies the conditions, the function will produce the given escape diagram. The conjugate method uses conjugacy to modify functions which have already known escape diagrams. The resulting function produces an escape diagram which is a distortion of the escape diagram of the known function. These methods were tested numerically, using computer programs to determine whether the functions that they result in produce the correct escape diagrams for multiple cases. Both of these methods were found to be successful in the cases tested.

1 Objective

An escape diagram is a plot which displays the set of all points whose orbit remains bounded under iteration of a given function. The objective of this project is to develop a method of finding a function which produces a given escape diagram. Two methods of accomplishing this have been developed in this endeavor.

2 Method 1: The Fixed Point Method

One method of finding a function which produces a given escape diagram utilizes repulsive and attractive fixed points. This will be referred to as the "fixed point" method. For simplicity, let us consider an escape diagram in the real numbers. Let this escape diagram be the set of points in the interval $[x_1, x_3]$, where $x_1 \neq x_3$, and let the unknown function which produces this escape diagram under iteration be a continuous function which maps the real numbers to the real numbers, $f : \Re \mapsto \Re$. Let x_1 and x_3 be repulsive fixed points, and let another point x_2 be an attractive fixed point which is between x_1 and x_3. Let x_1, x_2, and x_3 be the only fixed points under f. Let it also be the case that for all $x \in [x_1, x_3]$, $f(x) \in [x_1, x_3]$. This would make points in the interval $[x_1, x_3]$ be mapped only to points in the same interval $[x_1, x_3]$, and thus, their orbits will be bounded. Points outside of the interval $[x_1, x_3]$ will be repelled away from the interval, toward positive or negative infinity, and thus their orbits will not be bounded. In summary, a function which satisfies the following three conditions will produce an escape diagram which is the interval $[x_1, x_3]$:

1. x_1 and x_3 are repulsive fixed points, with an attractive fixed point x_2 between x_1 and x_3, that is, $f'(x_1) > 1$, $|f'(x_2)| < 1$, and $f'(x_3) > 1$.
2. There are no fixed points other than x_1, x_2, and x_3.
3. For all $x \in [x_1, x_3]$, $f(x) \in [x_1, x_3]$.

A reasonable candidate which could satisfy these conditions is a cubic function

$$f(x) = x + A(x - x_1)(x - x_2)(x - x_3)$$

where A is chosen to satisfy $f'(x_2) \in (0, 1)$ and x_2 is chosen to be the average of x_1 and x_3.

The fixed point method can be extended to escape diagrams in the plane, such that f is redefined as a continuous function which maps $\Re^2$ to itself

$[f : \Re^2 \mapsto \Re^2]$. For consistency with the next method, we choose to represent our function as a function on $\mathbf{C}^1$ rather than on $\Re^2$. This is because known escape diagrams for functions like z^2+c (where c is a constant) are written in complex coordinates. Since our definition of f will not be complex analytic, this is merely a convenient choice of coordinates. This can be done by letting x_1, x_2, and x_3 be functions of y, which is the imaginary component of the complex number z which is the argument of f $(z = x + iy)$.

$$x_1 \longrightarrow x_1(y)$$

$$x_2 \longrightarrow x_2(y)$$

$$x_3 \longrightarrow x_3(y)$$

Let the set of points in the escape diagram be a region R of the complex plane with a boundary which can be defined by, $x_1(y)$ and $x_3(y)$, with y in the interval (y_1, y_2) $[R = \{x + iy : x_1(y) \leq x \leq x_3(y), y_1 < y < y_2\}]$. This can be thought of as solving infinitely many of the real cases, one for each y-value in the interval (y_1, y_2).

The cubic function presented in the real case can be extended to the complex numbers with the function f being

$$f(z) = x + A(x - x_1(y))(x - x_2(y))(x - x_3(y)) + iy$$

Where A is chosen to satisfy $f'(x_2(y)) \in (0, 1)$ for all $y \in (y_1, y_2)$ and $x_2(y)$ is chosen to be the average of $x_1(y)$ and $x_3(y)$.

3 Method 2: The Conjugate Method

The second method involves conjugates of functions which have escape diagrams which are already known [1]. Suppose that the escape diagram of a given function g is known, and is the region R_g of the complex plane, such that $R_g = \{\rho e^{i\theta} : 0 \leq \rho \leq r(\theta), 0 \leq \theta < 2\pi\}$. Also suppose that the escape diagram of the unknown function f is a region R_f, and that R_g can be distorted into R_f using a homeomorphism h of the complex plane. f can then be expressed as $f = h \circ g \circ h^{-1}$.

Suppose the escape diagram of f is a region of the complex plane where the boundary can be expressed using a polar function $r = r(\theta)$, that is, $R_f = \{\rho_f e^{i\theta_f} : 0 \leq \rho_f \leq r(\theta_f), 0 \leq \theta_f < 2\pi\}$. Because it is already known that the escape diagram of the function z^2 is the unit disk, we can let R_g be the unit disk $[R_g = \{\rho_g e^{i\theta_g} : 0 \leq \rho_g \leq 1, 0 \leq \theta_g < 2\pi\}]$, and we can let g be defined by $g(z) = z^2$. We can also let h be defined by $h(z) = zr(\theta)$ where

$h^{-1}(z) = z/r(\theta)$ [θ can be expressed in terms of z as $\theta = \theta(z) = cos^{-1}\left(\frac{Re[z]}{|z|}\right)$ for $Im[z] > 0$ and $\theta = \theta(z) = -cos^{-1}\left(\frac{Re[z]}{|z|}\right)$ for $Im[z] < 0$]. f can therefore be expressed as

$$f(z) = (h \circ g \circ h^{-1})(z) = g\left(\frac{z}{r\left(\theta(z)\right)}\right) r\left(\theta\left(g\left(\frac{z}{r(\theta(z))}\right)\right)\right)$$

.

4 Tests and Examples

These methods were tested for several escape diagrams to find functions which produce them under iteration. For the fixed point method, the escape diagrams were required to have boundaries which can be expressed using the functions $x_1(y)$ and $x_3(y)$ as described in section 2. For the conjugate method, the escape diagrams were required to have boundaries that can be expressed using a polar function $r(\theta)$. For each escape diagram, a program was used which selected 10,000 random points in the complex plane, and iterated each point N times using the function that was found using one of the methods. If a point was within a distance R from the origin after N iterations, its orbit was considered to be bounded. If a point was beyond a distance R from the origin after N iterations, its orbit was considered to not be bounded. The points which had orbits which were considered to be bounded were plotted in black. The boundary of the intended escape diagram was plotted in red. The programming language that was used to write these programs was python.

4.1 Pseudo-code for Testing the Fixed Point Method

The escape diagram is given as a region of the complex plane that has a boundary which is defined by two given functions, x_1 and x_3, and a given interval (y_1, y_2). The pseudo-code for the program used in the fixed point method examples was the following:

1. input $x_1(y)$
2. input $x_3(y)$
3. define $x_2(y) = (x_1(y) + x_3(y))/2$
4. define $g(y) = |(x_2(y) - x_1(y))(x_2(y) - x_3(y)|$

5. set M to the maximum value of $g(y)$ on the interval (y_1, y_2)
6. set A to $1/M$
7. plot $x_1(y)$ and plot $x_3(y)$ for $y \in (y_1, y_2)$
8. for $i = 0$ to 10,000
   (a) set y to a random real number in the interval (y_1, y_2)
   (b) define $f(x) = x + A(x - x_1)(x - x_2)(x - x_3)$
   (c) set x_0 to a random real number in a given interval
   (d) set x to x_0
   (e) for $n = 0$ to 200
      i. set x to $f(x)$
   (f) if $\mid x \mid < R$
      i. plot (x_0, y)

4.2 Pseudo-code for Testing the Conjugate Method

The escape diagram is given as a region of the complex plane which has a boundary that is defined by a given polar function $r(\theta)$. The pseudo-code used for the program used for the conjugate method was the following:

1. define $\theta(z)$ as:
   (a) $cos^{-1}(Im[z]/|z|)$ if $Im[z] > 0$ (where y is the y-component of z)
   (b) $-cos^{-1}(Im[z]/|z|)$ if $Im[z] < 0$
2. input $r(\theta)$
3. define $g(z) = Re[z]^2 - Im[z]^2 + 2iRe[z]Im[z]$
4. define $h(z) = zr(\theta(z))$
5. define $h^{-1}(z) = z/r(\theta(z))$
6. define $f(z) = h(g(h^{-1}(z)))$
7. plot $(r(\theta)cos(\theta), r(\theta)sin(\theta))$ for the interval $-\pi \leq \theta \leq \pi$
8. for $i = 0$ to 10,000
   (a) set x_0 and y_0 to random numbers in a given interval

(b) set z_0 to $x_0 + iy_0$

(c) set z to z_0

(d) for $n = 0$ to 200

  i. if $z \neq 0$

    A. set z to $f(z)$

(e) if $|z| < R$

  i. plot (x_0, y_0)

4.3 Fixed Point Method Examples

In the first four examples of the fixed point method, the given escape diagrams where ellipses with a semi-major axis a and a semi-minor axis b. In the first four examples, let $x_1(y)$, $x_2(y)$, and $x_3(y)$ be defined on the interval $-b < y < b$ as

$$x_1(y) = -\sqrt{a^2\left(1 - \frac{y^2}{b^2}\right)} + \sqrt{a^2 - b^2}$$

$$x_2(y) = \sqrt{a^2 - b^2}$$

$$x_3(y) = \sqrt{a^2\left(1 - \frac{y^2}{b^2}\right)} + \sqrt{a^2 - b^2}$$

$x_1(y)$ and $x_3(y)$ can simply be derived from the equation for an ellipse in Cartesian coordinates, and $x_2(y)$ was chosen so that it vertically divides the ellipse down its center.

In the fifth example, let $x_1(y)$, $x_2(y)$, and $x_3(y)$ be defined on the interval $-2 < y < 2$ as

$$x_1(y) = y^2 - 4$$

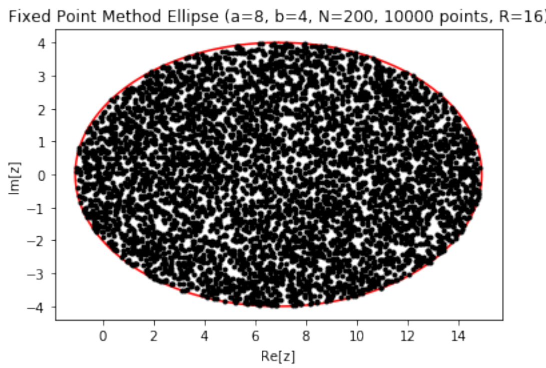
$$x_2(y) = (y^2 - 4 + sin^2(\pi y/2))/2$$

$$x_3(y) = sin^2(\pi y/2)$$

The function used for each example was

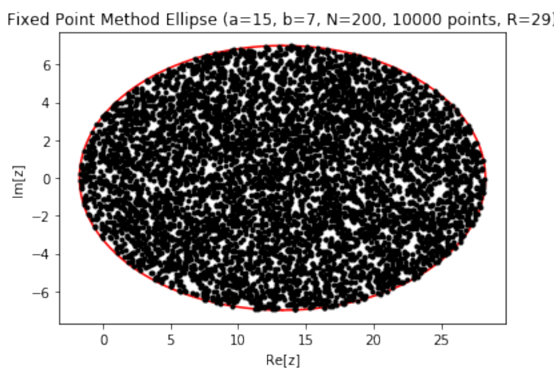
$$f(z) = x + A(x - x_1(y))(x - x_2(y))(x - x_3(y)) + iy$$

4.3.1 Fixed Point Method Example 1



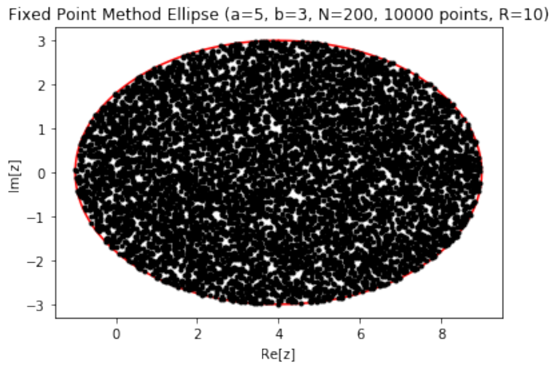
In this example, $a = 8$, $b = 4$, $N = 200$, and $R = 16$. The x-coordinates of the points were randomly chosen from the interval $-2 \leq x \leq 16$ and the y-coordinates from the interval $-5 < y < 5$.

4.3.2 Fixed Point Method Example 2



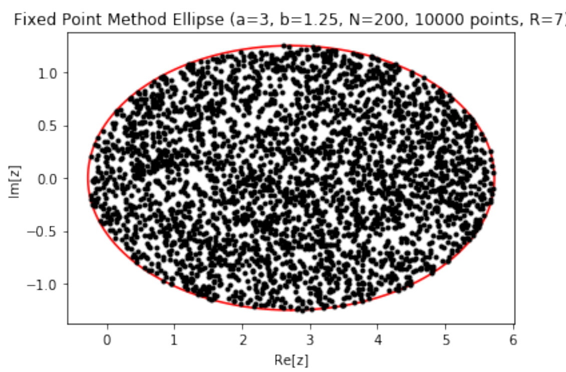
In this example, $a = 15$, $b = 7$, $N = 200$, and $R = 29$. The x-coordinates of the points were randomly chosen from the interval $-3 \leq x \leq 29$ and the y-coordinates from the interval $-8 < y < 8$.

4.3.3 Fixed Point Method Example 3



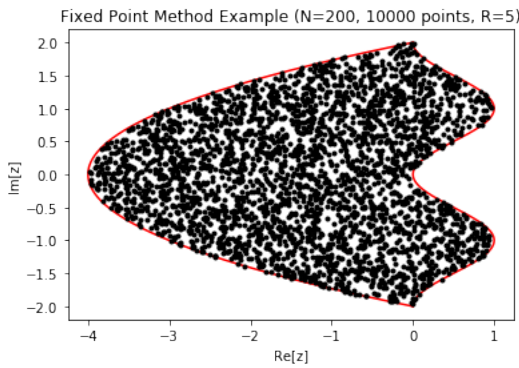
In this example, $a = 5$, $b = 3$, $N = 200$, and $R = 10$. The x-coordinates of the points were randomly chosen from the interval $-2 \leq x \leq 10$ and the y-coordinates from the interval $-4 < y < 4$.

4.3.4 Fixed Point Method Example 4



In this example, $a = 3$, $b = 1.25$, $N = 200$, and $R = 7$. The x-coordinates of the points were randomly chosen from the interval $-1 \leq x \leq 7$ and the y-coordinates from the interval $-2.25 < y < 2.25$.

4.3.5 Fixed Point Method Example 5



In this example, $N = 200$, and $R = 5$. The x-coordinates of the points were randomly chosen from the interval $-5 \leq x \leq 5$ and the y-coordinates from the interval $-2 < y < 2$.

4.4 Conjugate Method Examples

In the first four examples of the conjugate method, the given escape diagrams were ellipses which are identical to those in the respective fixed point method examples (where a is the semi-major axis and b is the semi-minor axis). In the first four examples,

$$r(\theta) = \frac{a(1-e^2)}{1-ecos(\theta)}$$

$$g(z) = z^2$$

$$h(z) = z\frac{a(1-e^2)}{1-eRe[z]/|z|}$$

$$h^{-1}(z) = \frac{z}{\left(\frac{a(1-e^2)}{1-eRe[z]/|z|}\right)}$$

where $e = \sqrt{1-(\frac{b}{a})^2}$ is the eccentricity. After substituting these into $f(z) = (h \circ g \circ h^{-1})(z)$ and simplifying, one arrives at the function

$$f(z) = \frac{z^2}{a\left(1-e^2\right)}\frac{1-e\left(\frac{Re[z]}{|z|}\right)}{1-e\left(\frac{Re[z^2]}{|z^2|}\right)}$$

In the fifth example,

$$r(\theta) = 1 + cos^2(\theta)$$

$$g(z) = z^2$$

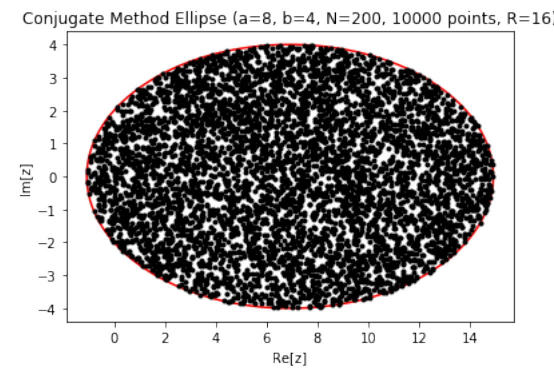
$$h(z) = z\left(1 + \left(\frac{Re[z]}{|z|}\right)^2\right)$$

$$h^{-1}(z) = \frac{z}{1 + \left(\frac{Re[z]}{|z|}\right)^2}$$

After substituting these into $f(z) = (h^{-1} \circ g \circ h)(z)$ and simplifying, one arrives at the function

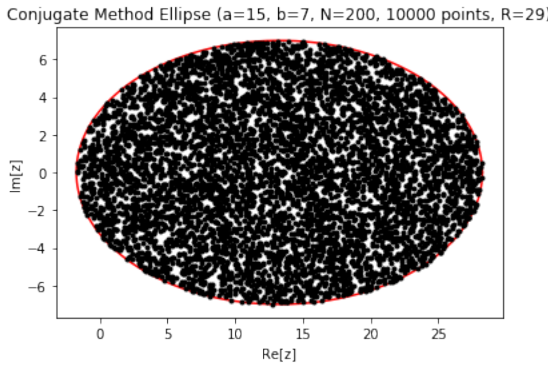
$$f(z) = z^2 \frac{1 + \left(\frac{Re[z^2]}{|z^2|}\right)^2}{\left(1 + \left(\frac{Re[z]}{|z|}\right)^2\right)^2}$$

4.4.1 Conjugate Method Example 1



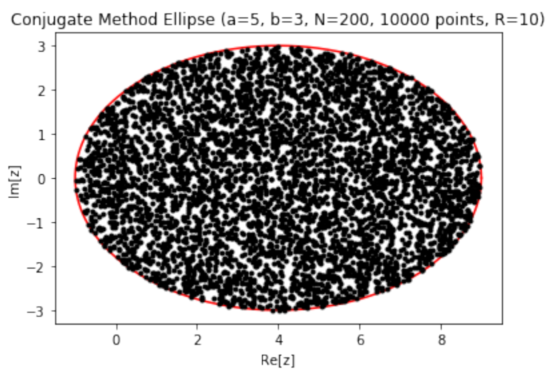
In this example, $a = 8$, $b = 4$, $N = 200$, $R = 16$. The x-coordinates of the points were randomly chosen from the interval $-2 \leq x \leq 16$ and the y-coordinates from the interval $-5 \leq y \leq 5$.

4.4.2 Conjugate Method Example 2



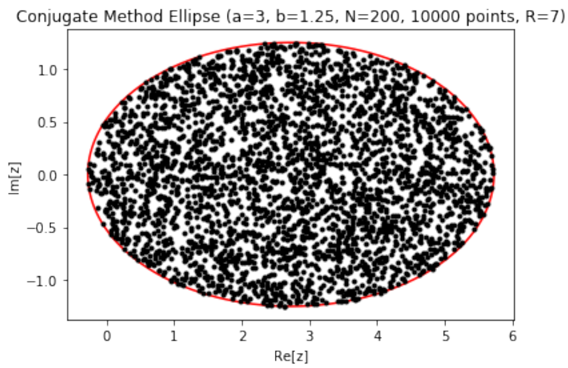
In this example, $a = 15$, $b = 7$, $N = 200$, $R = 29$. The x-coordinates of the points were randomly chosen from the interval $-3 \leq x \leq 29$ and the y-coordinates from the interval $-8 \leq y \leq 8$.

4.4.3 Conjugate Method Example 3



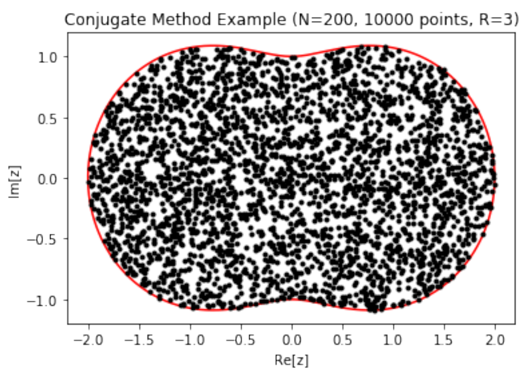
In this example, $a = 5$, $b = 3$, $N = 200$, $R = 10$. The x-coordinates of the points were randomly chosen from the interval $-3 \leq x \leq 10$ and the y-coordinates from the interval $-4 \leq y \leq 4$.

4.4.4 Conjugate Method Example 4



In this example, $a = 3$, $b = 0.5$, $N = 200$, $R = 7$. The x-coordinates of the points were randomly chosen from the interval $-1 \leq x \leq 7$ and the y-coordinates from the interval $-1.5 \leq y \leq 1.5$.

4.4.5 Conjugate Method Example 5



In this example, $N = 200$, $R = 7$. The x-coordinates of the points were randomly chosen from the interval $-3 \leq x \leq 3$ and the y-coordinates from the interval $-2 \leq y \leq 2$.

5 Conclusion

The objective of this research was to develop a method of finding a function which produces a given escape diagram. In this process, two methods were developed. The first method is called the fixed point method. It provides a set of conditions based on the positioning of fixed points, which guarantee that a function with those conditions produces the given escape diagram.

The second method is called the conjugate method. It utilizes conjugacy to distort escape diagrams of known functions to find functions which produce the given escape diagram. Both of these methods appear to be successful.

References

[1] Devaney, Robert L. A First Course in Chaotic Dynamical Systems: Theory and Experiment. Addison-Wesley, 1999.