A Study of Probability and Ergodic theory with applications to Dynamical Systems

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1 Introduction

I began this project by looking at a simple class of piecewise linear maps on the unit interval, and investigating the existence and properties of invariant ergodic measures corresponding to these dynamical systems. The class of linear maps I looked mostly at is a family of tent maps (to be defined later, but basically piecewise-linear unimodal maps on the unit interval). I began by numerical simulation in which I wrote software to indicate possible forms of the densities of natural invariant measures. By using the form of the density functions suggested by the numerical experiments, I computed some explicit formulas for the densities for a few special cases. Next I tried to generalize the existence of ergodic measures (to be defined later, but basically "irreducible" probability measures) in a class of tent maps. I tried several approaches to solve this problem. The ones that worked are explained in detail in this report and some of the more creative ideas I had that either didn't work or turned out later to be unnecessary are explained (and debunked in some cases) in the appendix.

In this report I will first explain the basic notions of probability that ergodic theory is founded on, then I will give a brief description of the ergodic theory I used and studied. Next, I will talk about the relevant dynamical systems theory used to analyze the Tent maps. Once all the theory is explained the problem I studied will be explained in detail and the general solution given. The general solution covers a class of functions much larger than the Tent maps; this is due to a 1980's result by Michal Misiurewicz. My proof of the existence and density of the ergodic measures for certain special cases of tent maps is the first section of the appendix. My proof does not apply to the full generality of functions that the Misiurewicz result does, it is rather specific to tent maps. Also in the appendix is an example of the procedure used for solving for the density functions of the invariant measures, as well as all my experimental results. The process for solving for the density function of the invariant measures does not come from the Misiurewicz proof.

The probability that I've studied for this project is a generalization of the ideas of probability as it is taught in the STAT 5571 class at the UMD. Primarily in this study, I used the textbook *Probability: Theory and Examples* by Richard Durrett. I studied the first two chapters on the basic theorems and definitions of probability (including expected value, independence, the Borel-Cantelli Lemmas, the weak LLN and the strong LLN, central limit theorems) and the sixth chapter in full (ergodic theory), and parts and portions of other chapters as was needed (chapter 4, which contains conditional expectation and martingales, chapter 5 which contains Markov chains, and the appendix, which contains a short course in the essentials of measure theory as developed using the Caratheodory extension theorem). I also had to study dynamical systems at a more advanced level than is offered in the MATH 5260 course, using mostly the text Chaotic Dynamical Systems by Robert Devaney. Most notably I studied the sections on symbolic dynamics, topological conjugacy, the Schwarzian derivative and the kneading sequence theory.

2 Notation and Theory of Probability

I will start off by listing some definitions needed for this paper; these are definitions and theorems from Probability theory that are more general or extensions of the concepts taught in introductory courses.

Definition 2.1. *σ-algebra: An algebra $\mathcal{F}$ is a collection of subsets of the nonempty sample space Ω that satisfy the properties,*

1. *if $A, B \in \mathcal{F}$ then $A \cup B \in \mathcal{F}$ and $A^c \in \mathcal{F}$*
2. *An algebra is a σ-algebra if for $A_i \in \mathcal{F}$ for $i \in \mathbb{Z}^+$, $\bigcup_{i=1}^{\infty} A_i \in \mathcal{F}$*

Definition 2.2. *A measure is a set function $\mu : \mathcal{F} \to [0, \infty)$ with the following properties:*

1. *$\mu(\emptyset) \leq \mu(A)$ for all $A \in \mathcal{F}$, where $\mu(\emptyset) = 0$*
2. *if $A_i \in \mathcal{F}$ is a countable or finite sequence of disjoint sets then $\mu(\bigcup_i A_i) = \sum_i \mu(A_i)$.*

A probability measure has $\mu(\Omega) = 1$, where Ω is usually called the sample space.

Definition 2.3. *A probability space is a triple $(\Omega, \mathcal{F}, \mu)$ where Ω is the sample space, or the set of outcomes, $\mathcal{F}$ is a σ-algebra of events, and μ is a probability measure.*

A special σ-algebra that will often be referenced, the Borel σ-algebra, denoted $\mathcal{B}$ is defined to be the smallest σ-algebra containing the collection of open sets.

Definition 2.4. *A function $X : \Omega \to \mathbb{R}$ is said to be a measurable map from $(\Omega, \mathcal{F})$ to $(\mathbb{R}, \mathcal{B})$ if $\{\omega : X(\omega) \in B\} \in \mathcal{F}$ for all $B \in \mathcal{B}$. In the future we will say $X^{-1}(B) \in \mathcal{F}$. $X^{-1}(B)$ is called the inverse image of B under X.*

Remark 2.5. *A function X is called a random variable if X is a measurable map from $(\Omega, \mathcal{F})$ to $(\mathbb{R}, \mathcal{B})$.*

Examples of probability measures are abundant,

Example 2.6. *We can define a probability measure on $[0, 1]$ using the density function for the uniform random variable: $f(x) = 1$ by $\mu(A) = \int_A dx$ for any measurable $A \subset [0, 1]$.*

Example 2.7. *Let $X : (\Omega, \mathcal{F}) \to (\mathbb{R}, \mathcal{B})$ be a random variable with density $f(x) = e^{-x}, x > 0$. We can define a probability measure by $\mu(A) = \int_A e^{-x} dx$, for $A \subset [0, \infty)$. Random variables with this distribution are called exponential random variables.*

A special measure that will be used often is the Lebesgue measure. Lebesgue measure, denoted by λ is the only measure for which $\lambda(A) = b - a$, where A is the interval (a, b), for any $a, b \in \mathbb{R}$ such that $a \leq b$. The derivation of this measure can be found in most measure theory texts, a classical derivation can by found in the text *Lebesgue Integration on Euclidean Space* by Frank Jones [6]. A less standard derivation appears in [1] by Durrett.

2.1 Expected Value

We now define the expected value of a random variable:

Definition 2.8. *The expected value of a random variable X with respect to a probability measure P is defined to be: $EX = \int_\Omega X dP$. We can also define $E(X; A) = \int_A X dP = \int X \mathbf{1}_A dP$.*

In the terms of introductory probability think of the dP as being the density function "$f(x)dx$" of the random variable. A formal definition of the integral dP can be found in any standard measure theory text, again, I refer the reader to [1] or [3].

Now we would like to define the concept of conditional expectation, that is to say, the expected value given that we know some events have occurred (where the information we know takes the form of a σ-algebra. We need some preliminary definitions first,

Definition 2.9. *The sigma-algebra generated by X is defined to be the smallest σ-algebra for which X is a measurable function, denoted $\sigma(X)$.*
$\sigma(X) = \{\{\omega : X(\omega) \in B\} : B \in \mathcal{B}\}$.

2.2 Conditional Expectation

Now we define conditional expectation:

Definition 2.10. *$E(X|\mathcal{F})$ is any random variable Y that satisfies:*

1. *$Y \in \mathcal{F}$, that is to say, Y is $\mathcal{F}$-measurable*
2. *$\int_A X dP = \int_A Y\, dP$ for all $A \in \mathcal{F}$.*

We say the conditional expectation of X given a random variable Y to be the expected value of X given the σ-algebra generated by Y, $E(X|Y) = E(X|\sigma(Y))$.

Example 2.11. *Say we have probability space $(\Omega, \mathcal{F}, P)$ and let X be a random variable on this space. We can define $E(X|\mathcal{F}) = X$, that is to say, if we have perfect knowledge for X, then the conditional expected value of X given all we know is X itself. Notice X certainly satisfies the conditions 1,2 of definition 2.10.*

Example 2.12. *Opposite of perfect information is no information: given the σ-algebra $\mathcal{F} = \{\Omega, \emptyset\}$, then $E(X|\mathcal{F}) = EX$. I.e. if we know nothing, then the best guess for the expected value of X is the expected value of X.*

A classic example from undergraduate probability is the following:

Example 2.13. *Let $\Omega = [0, \infty)$ and $\mathcal{F} = \mathcal{B}$ on $\mathbb{R}^+$. If X_1, X_2 are independent (defined below in section 2.3) exponential random variables with mean 1, i.e. the density $f(x) = e^{-x}$, let $S = X_1 + X_2$. $E(X_1|S)$ is a random variable Y that must satisfy (1) and (2) of definition 2.10. The joint density of (X_1, S) can be found using a transformation from the joint density of (X_1, X_2) to the joint density of $Y_1 = X_1$, $Y_2 = X_1 + X_2$. The Jacobian for this transformation is*

$$\mathbf{J} = \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix}$$

so

$$f_{Y_1,Y_2}(y_1, y_2) = f_{X_1,X_2}(y_1, y_2 - y_1)|\mathbf{J}| = e^{-y_1}e^{-(y_2-y_1)}(1) = e^{-y_2}$$

where $0 < y_1 < y_2$. The conditional density is then $f_{Y_1,Y_2}(y_1, y_2)/f_{Y_2}(y_2)$. Notice that Y_2 is the sum of two independent exponential random variables and hence has a gamma(1,2) distribution. The conditional distribution is

$$g(y_2) = \frac{e^{-y_2}}{y_2 e^{-y_2}} = \frac{1}{y_2}$$

Now,

$$E(X_1|S = y_2) = \int_0^{y_2} y_1 \frac{1}{y_2} dy_1 = \frac{y_2}{2}$$

where $y_2 > 0$. So we can say that $E(X_1|S) = S/2$. This is a very technical way to show what should be very intuitive given the definition of conditional expectation. We note that since X_1, X_2 are independent exponential random variables, $E(X_1; A) = E(X_2; A) = \int_A e^{-x} dx$ and it should be clear that $(X_1 + X_2)/2 \in \sigma(X_1 + X_2)$, so (1) is satisfied and $E(S/2; A) = (E(X_1; A) + E(X_2; A))/2 = E(X_1; A)$ so (2) is satisfied, hence $S/2 = E(X_1|S)$.

2.3 Independence

In the last example I used the concept of independence, so I will here define it.

Definition 2.14. *Let X, Y be random variables. If for all Borel sets A, B $P(X \in A, Y \in B) = P(X \in A)P(Y \in B)$ then X and Y are said to be independent.*

The idea of independence generalizes to σ-algebras:

Definition 2.15. *$\mathcal{F}$ and $\mathcal{G}$ are said to be independent σ-algebras if for all $A \in \mathcal{F}$ and $B \in \mathcal{G}$, A and B are independent events, that is $P(A \cap B) = P(A)P(B)$.*

The following propositions follow directly from the definition of independence and the definition of $\sigma(X), \sigma(Y)$:

Proposition 2.16. *If X, Y are independent then $\sigma(X), \sigma(Y)$ are independent. Furthermore, if $\mathcal{F}, \mathcal{G}$ are independent σ-algebras and if $X \in \mathcal{F}$ and $Y \in \mathcal{G}$, then X and Y are independent.*

A simple example of independence is:

Example 2.17. *Let X, Y be exponential random variables with means Θ_1, Θ_2 respectively. Say the joint density function for X, Y is*

$$f(x, y) = e^{-(\frac{x}{\Theta_1} + \frac{y}{\Theta_2})}$$

so we compute the probability,

$$P(X \in A, Y \in B) = \int_A \int_B e^{-(\frac{x}{\Theta_1} + \frac{y}{\Theta_2})}\, dxdy = \int_B e^{-\frac{y}{\Theta_2}} (\int_A e^{-\frac{x}{\Theta_1}}\, dx)\, dy = P(X \in A)P(Y \in B)$$

for any $A \in \sigma(X)$ and $B \in \sigma(Y)$, thus X and Y are independent.

2.4 Convergence

The two types of convergence most important to this report are convergence in probability and almost sure convergence.

Definition 2.18. *A sequence of random variables $\{X_n\}_{n=1}^{\infty}$ is said to converge in probability to X, $X_n \to X$ in prob. if for all $\epsilon > 0$, $P(|X_n - X| > \epsilon) \to 0$ as $n \to \infty$.*

Definition 2.19. *A sequence of random variables $\{X_n\}_{n=1}^{\infty}$ is said to converge almost surely to X, $X_n \to X$ a.s. if for all $\epsilon > 0$, $P(|X_n - X| > \epsilon \text{ i.o.}) = 0$ where i.o. stands for infinitely often. This is to say that $X_n \to X$ except possibly on a set of measure (probability) 0.*

The following proposition follows directly from the above definitions:

Proposition 2.20. *Let $\{X_n\}_{n=1}^{\infty}$ be a sequence of r.v.'s s.t. $X_n \to X$ a.s. as $n \to \infty$, then $X_n \to X$ in prob. as $n \to \infty$.*

I'll now state two theorems which deal with the convergence of sums of random variables, the weak and the strong law of large numbers (as written in [1]).

Theorem 2.21. *Let $X_1, X_2, \ldots$ be independent and identically distributed, and let $S_n = X_1 + \ldots + X_n$. In order that there exist constants μ_n so that $S_n/n - \mu_n \to 0$ in probability, it is necessary and sufficient that*

$$xP(|X_1| > x) \to 0 \text{ as } x \to \infty.$$

We can take $\mu_n = E(X_1 1_{(|X_1| \leq n)})$.

Now a stronger version of the above,

Theorem 2.22. *Strong Law of Large Numbers: Let $X_1, X_2, \ldots$ be i.i.d. r.v.'s with $E|X_i| < \infty$. Let $EX_i = \mu$ and $S_n = X_1 + \ldots + X_n$. Then $S_n/n \to \mu$ a.s. as $n \to \infty$.*

3 Ergodic Theory

Now, the basic definitions being out of the way, we proceed to the ergodic theory which has been the main focus of this project. Intuitively, ergodic theory is concerned with taking certain (stationary) sequences and saying something about the convergence of the average of these sequences. If you have a function $f : \mathbb{R} \to \mathbb{R}$ and a (stationary) sequence $\{X_m\}_{m \geq 0}$, then under what conditions can you say

$$\lim_{n\to\infty} \frac{1}{n} \sum_{m=0}^{n-1} f(X_m)$$

exists? From the strong law of large numbers we know that if the sequence is composed of independent and identically distributed (iid) random variables with $E|X_1| < \infty$, then

$$\lim_{n\to\infty} \frac{1}{n} \sum_{m=0}^{n-1} X_m = \mu = EX_i \ a.s.$$

The ergodic theorem is a sort of generalization of the SLLN. It states that if we impose some additional structure on $\{X_m\}_{m \geq 0}$, namely that the sequence is stationary and $E|f(X_0)| < \infty$ then

$$\lim_{n\to\infty} \frac{1}{n} \sum_{m=0}^{n-1} f(X_m) \text{ exists } a.s.$$

If the sequence has the additional property of being ergodic, then

$$\lim_{n\to\infty} \frac{1}{n} \sum_{m=0}^{n-1} f(X_m) = Ef(X_0) \ a.s.$$

Now formal definitions for the terms above.

Definition 3.1. *Stationary sequence: A sequence* $\{X_m\}_{m\geq 0}$ *is said to be stationary if* $P((X_0, X_1, \ldots, X_m) \in A) = P((X_k, X_{k+1}, \ldots, X_{k+m}) \in A)$ *for all* $m, k \geq 0$ *and* $A \in \mathcal{B}^{m+1}$. *We can say the distribution of* X_n *is the same as the shifted distribution for any shift value of* k.

I have looked largely at maps that have the property that they are measure preserving.

Definition 3.2. *A map* $\phi : (\Omega, \mathcal{F}) \to (\Omega, \mathcal{F})$ *is said to be measure preserving with respect to a probability measure* P *if* $P(\phi^{-1}A) = P(A)$ *for all* $A \in \mathcal{F}$. *That is the measure of the inverse image of a set is the same as the measure of the set.*

Now I state a theorem (proof due to Durrett [1]):

Theorem 3.3. *If* $X \in \mathcal{F}$ *and* ϕ *is measure preserving, then* $X_n(\omega) = X(\phi^n(\omega))$ *defines a stationary sequence.*

Proof. To see why this theorem is true, let $B \in \mathcal{B}^{n+1}$ and define $A = \{\omega : (X_0(\omega), X_1(\omega), \ldots, X_n(\omega)) \in B\}$, then

$$\begin{aligned} &P[(X_k(\omega), X_{k+1}(\omega), \ldots, X_{k+n}(\omega)) \in B] \\ &= P[(X(\phi^k\omega), X(\phi^{k+1}\omega), \ldots, X(\phi^{k+n}\omega)) \in B] \\ &= P(\phi^k\omega \in A) = P(\omega \in A) = P(X_0, X_1, \ldots, X_n) \in B \end{aligned}$$

where the second to last equality is due to ϕ being measure preserving.

Hence $X_n(\omega)$ is stationary. □

Another important concept in probability is the idea of invariance.

Definition 3.4. *A set* A *is* **invariant** *if* $\phi^{-1}A = A$, *where equality is defined if the symmetric difference has measure 0:* $\mu[(\phi^{-1}A - A) \cup (A - \phi^{-1}A)] = 0$.

We can define the σ-algebra generated by the class of invariant events.

Proposition 3.5. *The class of invariant events is a* σ*-algebra, denoted* $\mathcal{I}$.

Proof. If $A, B \in \mathcal{I}$, consider $A \cup B$:

$$\begin{aligned} \phi^{-1}(A \cup B) &= \{\omega : \phi(\omega) \in A \cup B\} \\ &= \{\omega : \phi(\omega) \in A\} \cup \{\omega : \phi(\omega) \in B\} \\ &= \phi^{-1}(A) \cup \phi^{-1}(B) = A \cup B \end{aligned}$$

Now we look at complements,

$$\begin{aligned} \phi^{-1}A^c &= \{\omega : \phi(\omega) \in A^c\} \\ &= \{\omega : \phi(\omega) \in A\}^c \\ &= A^c \text{ as above.} \end{aligned}$$

Similar for countable unions. □

A σ-algebra connected to the class of invariant events is the tail σ-field:

Definition 3.6. *Consider a sequence of random variables $X_1, X_2, \ldots$, a tail event is an event whose occurrence or failure is determined by the sequence but is independent from any finite subsequence of these random variables. Formally, let $\mathcal{F}_n = \sigma(X_n, X_{n+1}, \ldots)$ then the tail σ-field is $\mathcal{T} = \bigcap_n \mathcal{F}_n$.*

Connected with $\mathcal{T}$ is the following useful result:

Theorem 3.7. *Kolmogorov's 0-1 Law: If $X_1, X_2, \ldots$ are independent, then for any $A \in \mathcal{T}$, $P(A) = 0$ or 1.*

We now have everything necessary to define what it means for a transformation or a stationary sequence to be ergodic.

Definition 3.8. *A measure preserving transformation ϕ is said to be ergodic if $\mathcal{I}$ is trivial, that is if for any $A \in \mathcal{I}$, then $P(A) = 0$ or 1.*

An ergodic transformation is one where the only invariant events are almost all the points, or almost none of the points. For a stationary sequence to be ergodic, the transformation associated with it must be ergodic.

Remark 3.9. *Recall*

$$X_n(\omega) = X(\phi^n(\omega)) \tag{1}$$

defines a stationary sequence, so if ϕ is ergodic, then $\{X_m\}_{m \geq 0}$ is ergodic.

Remark 3.10. *We consider only stationary sequences induced by a measure preserving map ϕ as in* (**??**).

Proposition 3.11. *All stationary sequences are formed as above.*

Proof. Let $Y_0, Y_1, \ldots$ be a stationary sequence taking values in a nice space $(S, \mathcal{L})$. We use the Kolmogorov extension theorem to construct a probability measure P on the sequence space $(S^{\{0,1,\ldots\}}, \mathcal{L}^{\{0,1,\ldots\}})$ so that $X_n(\omega) = \omega_n$ has the same distribution as Y_n. Take ϕ to be the shift operator, $\phi(\omega_0, \omega_1, \ldots) = \phi(\omega_1, \omega_2, \ldots)$ and say $X(\omega) = \omega_0$, then ϕ is measure preserving and $X_n(\omega) = X(\phi^n \omega)$. $\square$

We draw attention to an important observation:

Remark 3.12. *Let $\Omega = \mathbb{R}^{\{0,1,\ldots\}}$ and ϕ be the shift operator. Let X be a random variable in $(\Omega, \mathcal{B}^{\{0,1,\ldots\}}, P)$, recall the stationary sequence $X_1, X_2, \ldots$ where $X_n(\omega) = X(\phi^n \omega)$. Note that an invariant set under the shift operator by definition has $A = \{\omega : \omega \in A\} = \{\omega : \phi\omega \in A\}$. Hence $A \in \sigma(X_1, X_2, \ldots)$. We observe now that $\{\omega : \phi\omega \in A\} = \{\omega : \phi^2\omega \in A\}$ from which we deduce that $A \in \sigma(X_1, X_2, \ldots) \cap \sigma(X_2, X_3, \ldots)$. Iteration allows us to conclude that $A \in \bigcap_n \sigma(X_n, X_{n+1}, \ldots)$ which is exactly $\mathcal{T}$. This gives us the useful result that for any $A \in \mathcal{I}$, that $A \in \mathcal{T}$ as well, hence $\mathcal{I} \subset \mathcal{T}$.*

Now we get to the main result about ergodic sequences.

Theorem 3.13. *Birkhoff Ergodic Theorem: If ϕ is a measure preserving transformation on the probability space $(\Omega, \mathcal{F}, P)$, then for any $X \in L^1(\Omega)$,*

$$\frac{1}{n}\sum_{m=0}^{n-1} X(\phi^m \omega) \to E(X|\mathcal{I}) \text{ a.s. and in } L^1.$$

Remark 3.14. *If the transformation ϕ is ergodic, then the average will converge to EX since $\mathcal{I}$ is trivial when ϕ is ergodic.*

We can recover the SLLN from the ergodic theorem. Let $X_1, X_2, \dots$ be an i.i.d. sequence of random variables. Kolmogorov's 0-1 law implies that $\mathcal{T}$ is trivial, and since $\mathcal{I} \subset \mathcal{T}$ then $\mathcal{I}$ is trivial. Hence $\{X_i\}_i$ is ergodic and the ergodic theorem gives:

$$\frac{1}{n}\sum_{m=0}^{n-1} X_m(\omega) \to E(X_0) \text{ a.s. and in } L^1.$$

So, as claimed earlier, the ergodic theorem can be thought of as a generalization of the SLLN. Along with these tools, we need some theorems and definitions from Dynamical Systems theory.

4 Dynamical Systems Theory

The application I studied in this project was a very simple dynamical system which exhibits very interesting long-term behavior, particularly, chaotic behavior. To explain the problem coherently, we will need some new terminology.

Definition 4.1. *Absolutely Continuous Measure: A measure ν is absolutely continuous with respect to measure μ, written $\nu << \mu$, if $\mu(A) = 0 \Rightarrow \nu(A) = 0$.*

All measures should from this point forward be considered absolutely continuous with respect to Lebesgue measure unless otherwise stated.

Definition 4.2. *Critical Point: a critical point of a map $\phi : \Omega \to \mathbb{R}$ is any $x \in \Omega$ s.t. $\phi'(x) = 0$ or does not exist.*

Definition 4.3. *Orbit: the orbit of a point a under the map ϕ is the sequence $\{\phi(a), \phi^2(a), \dots\}$*

Definition 4.4. *Fixed Point: a fixed point of a map ϕ is any point that satisfies $\phi(a) = a$.*
A Periodic point of period n is any point that satisfies $\phi^n(a) = a$.

Definition 4.5. *Attracting Fixed/Period point: if for a map ϕ and a fixed point a, $|\phi'(a)| < 1$, then a is called an attracting fixed point. Similarly, for a periodic point a of period n, if $|(\phi^n)'(a)| < 1$, then the cycle $(a, \phi(a), \phi^2(a), \dots, \phi^{n-1}(a))$ is called a stable or attracting cycle of period n.*

Remark 4.6. *The orbit of a critical point is important and is usually called the kneading sequence.*

We will later state and prove theorems concerning these ideas, but for now, we state the application studied during this project.

5 An Application of Ergodic theory to a Dynamical System

The problem that was considered in this project concerns a family of maps on the unit interval called tent maps. These maps are defined as:

$$\phi_{\alpha,h}(x) = \begin{cases} \frac{h}{\alpha}x & \text{if } 0 \leq x < \alpha \\ \frac{h}{1-\alpha}(1-x) & \text{if } \alpha \leq x < 1 \end{cases} \tag{2}$$

$\alpha, h : h/\alpha > 1,\ h/(1-\alpha) > 1$ where α, h are parameters that represent the position of the critical point and the height of the map.

The focus in the analysis to follow is to determine for which values of h and α the tent map associated with those parameters is ergodic (with respect to some invariant measure that is absolutely continuous with respect to Lebesgue measure). The ergodicity of these maps indicate a property of the long-term behavior of these maps under iteration. The determination of the long-term behavior of this map is of general interest. My other interest is in finding the invariant ergodic measures.
A special case of this map is when $\alpha = 1/2$ and $h = 1$. This gives what can be called "the full height" symmetric tent map. The value of 1 for h represents a division in the dynamic behavior of these maps. When $h > 1$, the iteration of the map no longer is contained in the interval $[0, 1]$; the map must be extended to the entire real line in this case. All points except for an invariant Cantor set will escape to infinity in this case. When $h < 1$, there will be an invariant interval contained in $[0, 1]$. In the so-called "full height" case $h = 1$, the entire unit interval is invariant and the map $\phi_{\alpha,1}$ will preserve Lebesgue measure. Figure 1 shows an example of a tent map, this tent map has a finite critical orbit ($\phi^2(\phi(1/2))$ is a fixed point) and is symmetric.

Proposition 5.1. *The full height tent map, $\phi_{\alpha,1}$ preserves λ on [0,1].*

Proof. It suffices to prove the result for intervals $(a, b) \subset [0, 1]$. Let $0 \leq a < b < 1$ be given. Notice the pre-image of (a, b) under $\phi_{\alpha,1}$ consists of two intervals: $(a\alpha, b\alpha) \cup (1-b(1-\alpha), 1-a(1-\alpha))$. Now we take the measure of the pre-image:

$$\begin{aligned} &\lambda[(\phi^{-1}_{\alpha,1}(a, b)] \\ = \lambda[(a\alpha, b\alpha)] + \lambda&[(1-b(1-\alpha), 1-a(1-\alpha))] \\ &= \alpha(b-a) + (1-\alpha)(b-a) = b-a \end{aligned}$$

The case $b = 1$ is the same really, the pre-image can be thought of as 2 intervals sharing the common endpoint α. The result is the same. Hence, Lebesgue

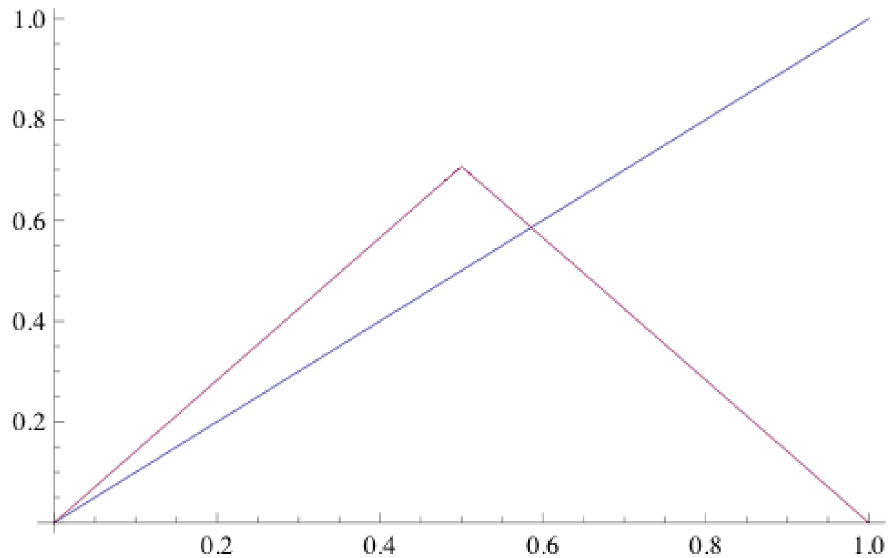

Figure 1: $\phi_{1/2,1/\sqrt{2}}(x)$

measure is preserved for all open intervals contained in $[0,1]$ and hence for all Borel sets $B \in \mathcal{B} \cap [0,1]$. $\square$

We restrict attention to the specific case $\alpha = 1/2$. This case is the full height symmetric tent map on $[0,1]$.

Now we need to show that $\mathcal{I}$ is trivial.

Proposition 5.2. *The σ-algebra of invariant events is trivial for $\phi_{\frac{1}{2},1}$.*

Proof. We use Fourier analysis for this result. We know that if f is measurable on [0,1] and f is square-integrable ($\int_{[0,1]} |f(x)|^2 \, dx < \infty$) or $f \in L^2([0,1])$, then f has a unique Fourier expansion: $f(x) = \sum_k c_k e^{2\pi ikx}$ where equality is convergence of the partial sums

$$\sum_{k=-K}^{K} c_k e^{2\pi ikx} \to f(x) \text{ in } L^2[0,1]$$

The coefficients $c_k = \int_0^1 f(x) e^{-2\pi ikx} \, dx$ are unique.
Now, take f to be the indicator function on some invariant set $A \in \mathcal{I}$, $f(x) = 1_A(x)$. Assume $\lambda(A) > 0$. Notice that for $x \in A$, $f(x) = f(T(x))$ a.e. Since 1_A is measurable and square-integrable, we have a Fourier expansion for 1_A, say

$$1_A(x) = \sum_{k=-\infty}^{\infty} c_k e^{2\pi ikx}.$$

Now, suppose $x \in A \cap [0, \frac{1}{2})$, then

$$1_A(x) = 1_A(2x) = \sum_k c_k e^{2\pi ik2x} = \sum_k c_k e^{2\pi ikx} e^{2\pi ikx}$$

$$\Rightarrow c_k = c_k e^{2\pi ikx} \quad \text{for all } x \in A \cap [0, 1/2)$$
$$\Rightarrow c_k = 0 \text{ for all } k \neq 0$$

and if $x \in A \cap [1/2, 1]$ then

$$1_A(x) = 1_A(2(1-x)) = \sum_k c_k e^{2\pi ik(2(1-x))} = \sum_k c_k e^{-6\pi ikx} e^{2\pi ikx}$$

$$\Rightarrow c_k = c_k e^{-6\pi ikx} \text{ for all } x \in A \cap [1/2, 1] \text{ and } k \neq 0$$
$$\Rightarrow c_k = 0 \text{ for all } k \neq 0$$

Hence, $c_k = 0$ for $k \neq 0$, which means that $f(x) = 1_A(x)$ is constant on $[0, 1]$. This shows that either $1_A(x) = 0$ for all $x \in [0, 1]$ which implies $A = \emptyset$ a.e. or $1_A(x) = 1$ on $[0, 1]$ which implies $A = [0, 1]$ a.e. due to our assumption that $\lambda(A) > 0$. Since $A \in \mathcal{I}$ is arbitrary, this shows that $\mathcal{I}$ is trivial.

$\square$

This proves directly that the full height tent map on $[0, 1]$ is ergodic with respect to Lebesgue measure. This direct approach works only for the full height symmetric tent map. We will now demonstrate that for a family of tent maps including the full height map, that the tent map is equivalent to the shift and flip operator on the sequence space $\{0, 1\}^{\{0,1,\ldots\}}$. This conjugacy will allow us to conclude that $\mathcal{I}$ is trivial.

Proposition 5.3. *$\phi_{1/2,1}(x) = T(x)$ is equivalent to the shift and flip operator τ on the sequence space above, denoted Σ. Equivalent is in the sense that there is a homeomorphism $h : [0, 1] \to \Sigma$ such that $h \circ T = \tau \circ h$. We define τ as,*

$$\tau(\omega_0, \omega_1, \ldots) = \begin{cases} (\omega_1, \omega_2, \ldots) & \text{if } \omega_0 = 0 \\ (1-\omega_1, 1-\omega_2, \ldots) & \text{if } \omega_0 = 1 \end{cases}$$

Proof. We use the binary representation of $x \in (0, 1)$. The homeomorphism, h, between $[0, 1]$ and Σ is provided by the binary representation $h(x)$ for $x \in [0, 1]$. Let $x \in [0, 1]$ have to following binary expansion $x = \sum_{m=0}^{\infty} a_m/2^{n+1}$ where $a_m = 0$ *or* 1. Then $h(x) = (a_0, a_1, a_2, \ldots)$.
As a map this is not well defined for all x since for any dyadic number, $h(x)$ is not unique. For example, 1/2 can be represented as either .01111... or .100000... depending on which of the two intervals 1/2 is thought to lie in. To deal with this discrepancy, let $I_0 = [0, 1/2)$ and $I_1 = [1/2, 1]$. We show that the action of T on [0,1] is the same then as τ on Σ: let $h(x) = (a_0, a_1, \ldots)$ with $a_i \in \{0, 1\}$. If $a_0 = 0$ then

$$h \circ T(x) = h(2x) = h\left(2 \sum_{m=0}^{\infty} \frac{a_m}{2^{m+1}}\right) = h\left(a_0 + \sum_{m=0}^{\infty} \frac{a_{m+1}}{2^{m+1}}\right) = (a_1, a_2, \ldots) = \tau(h(x))$$

If $a_0 = 1$ then

$$h \circ T(x) = h(2(1-x)) = h\left(2\left[1 - a_0 + \sum_{m=1}^{\infty} \frac{1-a_m}{2^{m+1}}\right]\right) = h\left(\sum_{m=0}^{\infty} \frac{1-a_{m+1}}{2^{m+1}}\right)$$

$$= (1-a_1, 1-a_2, \ldots) = \tau(h(x))$$

We see that T is equivalent to the shift and flip map on the sequence space. $\square$

This conjugacy is the key step to proving the ergodicity of T. To complete the proof, we would need to show that τ is ergodic, and that topological conjugacies preserve ergodicity. We do not show here that τ is ergodic, but we do show at (**??**) that topological conjugacy preserves ergodicity.

The full height symmetric tent map is very nice in that we have this very clean proof for it's ergodicity. We have taken advantage of expressing the transformation in terms of binary coefficients (that we know exist) for every number in [0,1]. For this case alone we have the equality $\frac{h}{\alpha} = \frac{h}{1-\alpha}$ as well. I was unsuccessful in generalizing this technique to the case when the map is not symmetric or not full height. Once either symmetry or full height is lost, the binary representation of x and the tent map as an operator on the symbol space is lost. However, there does exist a general theory developed by Dr. Misiurewicz that established the ergodicity of many tent maps (and applies to a wider range of functions).

6 Solution for a more General Class

We now turn our attention back to the general family of tent maps defined in equation (**??**) of section 5.

Theorem 6.1. *If the kneading sequence of the map is finite then the class of tent maps as defined in* (**??**) *is ergodic.*

This limits our attention to the parameters choices of h and α for which the critical orbit is finite (that is to say it lands on a repelling fixed or periodic point after a finite amount of iterations).
To prove this theorem we refer to a theory developed by Michal Misiurewicz ([2]). This theory deals with a broad class of maps on the interval and shows ergodic properties of these maps. We will state first which maps this theory applies to, then show that tent maps satisfy the conditions (or rather restrict our attention on tent maps which satisfy the conditions), then state the theorem which gives existence of invariant ergodic measures on an invariant interval. We next show the construction for the density of the ergodic measures and provide an example of the process.
We need more definitions from dynamical systems theory to begin with,

Definition 6.2. *Schwarzian Derivative:* $Sf(x) = \frac{f'''(x)}{f'(x)} - \frac{3}{2}\left(\frac{f''(x)}{f'(x)}\right)^2$. *The Schwarzian derivative contains important properties of the dynamics of a map.*

The sign of the Schwarzian derivative is an indication to the existence of attracting periodic points and in some cases, how many attracting periodic points there can be in a map.

This theory applies to maps on a closed interval I with the following properties: let A be a finite subset $A \subset I$ containing its endpoints. We consider maps $f : I \setminus A \to I$ that are continuous and strictly monotone on components of $I \setminus A$. Furthermore, we require that f satisfy the following:

1. f is of the class C^1 and Lipshitz
2. $f' \neq 0$
3. $Sf \leq 0$
4. If $f^p(x) = x$, then $(f^p)'(x) > 1$
5. There exists a neighborhood U of A such that for every $a \in A$ and $n \geq 0$, $f^n(a) \in A \cup (I \setminus U)$
6. For every $a \in A$ there exists constants $\delta, \alpha, \omega > 0$ and $u \geq 0$ such that: $\alpha|x - a|^u \leq |f'(x)| \leq \omega|x - a|^u$ for every $x \in (a - \delta, a + \delta)$.

We now show that our tent maps lie in this class of functions. The closed interval we consider is [0,1], or closed intervals contained in [0,1]. I will be the invariant interval for the map $\phi_{\alpha,h}$. The endpoints for the invariant interval are determined by the kneading sequence for the map. The right endpoint is $\phi_{\alpha,h}(\alpha)$ and the left is $\phi^2_{\alpha,h}(\alpha)$ where α is the critical point. Since we want an invariant interval we look only at maps with parameters $0 < \alpha < 1$, and $0 < h < 1$, the extremal cases being less interesting. Our finite subset $A \subset I$ consists of the endpoints of I and the critical point, $A = \{\frac{h(1-h)}{1-\alpha}, h, \alpha\}$.

1. All tent maps are C^1 on the components of $I \setminus A$, they are C^∞ actually.
2. $\phi'_{\alpha,h}(x) \neq 0$ for all $x \in I \setminus A$.
3. The Schwarzian derivative is zero everywhere on $I \setminus A$.
4. In order for condition (4) to be satisfied, we require that $h > \alpha$ so the magnitude of the derivative of $\phi_{\alpha,h}$ is greater than 1.
5. Condition (5) requires that iterates of the critical point and iterates of the endpoints either land on a point in A or stay away from points in A. This condition is satisfied if we look only at cases where the orbit of the critical point is fixed after a finite number of iterations. The orbits of the endpoints follow the orbit of the critical point (follows from our definition of the endpoints).

6. Condition (6) is trivially satisfied by $\phi_{\alpha,h}$ since $|\phi'_{\alpha,h}| \leq \max\{\frac{h}{\alpha}, \frac{h}{1-\alpha}\}$ for all $x \in I \setminus A$.

   All tent maps will satisfy conditions (1), (2), (3), (6), and whenever $h > \alpha$ condition (4). The only condition not satisfied *a priori* by all tent maps with non-trivial dynamics and non-zero measure invariant sets is condition (5). It is for this reason only that attention has been restricted to maps with a finite kneading sequence. Experiments indicate that if the kneading sequence does not contain a finite number of points, then the natural measure associated with the mapping is not absolutely continuous with respect to Lebesgue measure (more on this later).

Example 6.3. *An example of a tent map which has a finite kneading sequence can be easily constructed. Take the map*

$$\phi_{\frac{1}{\sqrt{2}},\frac{1}{2}}(x) = \begin{cases} \sqrt{2}x & 0 \leq x < \frac{1}{2} \\ \sqrt{2}(1-x) & \frac{1}{2} \leq x \leq 1 \end{cases}$$

This tent map has the special property that the critical orbit is fixed after 2 iterations: $\phi(\frac{1}{2}) = \frac{\sqrt{2}}{2}$, $\phi^2(\frac{1}{2}) = \sqrt{2}(1-\frac{\sqrt{2}}{2}) = \sqrt{2}-1$, $\phi^3(\frac{1}{2} = \sqrt{2}(\sqrt{2}-1) = 2-\sqrt{2} = \frac{\sqrt{2}}{\sqrt{2}+1}$. *One can easily calculate that the fixed point is* $\frac{\sqrt{2}}{\sqrt{2}+1}$. *The invariant interval is* $I = [\sqrt{2}-1, \frac{1}{\sqrt{2}}]$. *A precise graph of this map illustrating the finite kneading sequence is displayed in figure 1.*

6.1 Existence of Ergodic Measures

The main theorem for the existence of ergodic measures absolutely continuous with respect to Lebesgue measure is,

Theorem 6.4. *Let f satisfy (1) - (6), then there exist probability f-invariant measures $\mu_1, \mu_2, \ldots, \mu_s$ absolutely continuous with respect to Lebesgue measure and a positive integer k such that:*

(a) supp $\mu_i = \overline{\bigcup_i G_i}$ for certain equivalence classes G_i of the relation $\approx$ for $i = 1, 2, \ldots, s$

(b) supp $\mu_i \cap$ supp μ_j is a finite set if $i \neq j$

(c) $1 \leq s \leq$ Card $A - 2$

(d) μ_i is ergodic for $i = 1, 2, \ldots, s$

(e) $\frac{d\mu_i}{d\lambda} \in \mathcal{D}_0$ for $i = 1, 2, \ldots, s$

(f) $\inf_V\{\frac{d\mu_i}{d\lambda}\} > 0$ for $i = 1, 2, \ldots, s$ where $V =$ supp $\mu_i \setminus B$

(g) if $\rho \in L^1(\lambda)$ then $\lim_{n\to\infty} \sum_{j=1}^{k} f^{n+j}(\rho) = \sum_{i=1}^{s} \alpha_i \frac{d\mu_i}{d\lambda}$ in $L^1(\lambda)$ where $\alpha_i = \int_D \rho \, d\lambda$ where $D = \bigcup_{n=0}^{\infty} f^{-n}($supp $\mu_i)$. If ρ is continuous then the convergence is also in the topology of uniform convergence on compact sets.

(h) *for every finite Borel measure ν that is absolutely continuous with respect to λ and f-invariant, one has $\nu = \sum_{i=1}^{s} \alpha_i \mu_i$, where $\alpha_i = \nu(\bigcup_{n=0}^{\infty} f^{-n}(supp\ \mu_i))$.*

For unimodal maps (such as tent maps), $Card\, A - 2 = 1$, since A has two endpoints and one critical point. Hence for each tent map satisfying (1) - (6) there is a unique $\phi_{\alpha,h}$-invariant measure absolutely continuous with respect to Lebesgue measure.
In part (e) the space $\mathcal{D}_0$ was referred to but not explained, we now elaborate on this result.

Definition 6.5. *For the open subset $U \subset I$ consisting of a finite number of intervals such that the endpoints of I belong to U, denote by B a subset of $I \setminus U$ such that $f(B) \subset B$. Denote by $\mathcal{D}_0(J)$ the set of all C^0 positive functions τ on J such that $\frac{1}{\sqrt{\tau}}$ is concave.*

So, $\mathcal{D}_0$ is the set of all functions τ on B such that $\tau|_J \in \mathcal{D}_0(J)$ for all components J of B. $\mathcal{D}_0$ is called the topology of uniform convergence on compact sets.
The notation $\frac{d\mu_i}{d\lambda}$ is a Radon-Nikodym derivative. To explain its meaning we state a theorem:

Theorem 6.6. *(Radon-Nikodym) If μ, ν are σ-finite measures and $\nu << \mu$ then there is a $g \geq 0$ such that $\nu(E) = \int_E g\, d\mu$. If h is another such function, then $g = h$ μ almost surely.*

The function g is called the Radon-Nikodym derivative and is denoted $g = \frac{d\nu}{d\mu}$. Then (e) can be interpreted that the Radon-Nikodym derivative of μ, our unique measure (that has $\mu << \lambda$) with respect to λ is in $\mathcal{D}_0$, which means that the density function of μ is continuous and positive on components of B, where in our case B is $I \setminus U$ with U being an open set consisting in neighborhoods of the points in A.
This theorem then shows us existence and uniqueness of $\phi_{\alpha,h}$ invariant measures for parameters (α, h) chosen so that the sequence $\{\phi_{\alpha,h}(\alpha), \phi^2_{\alpha,h}(\alpha), \phi^3_{\alpha,h}(\alpha), \ldots\}$ consists of a finite number of distinct points.
While this theorem provides us with existence and uniqueness of these measures, neither the theorem nor its proof gives us a way of constructing these measures. One can construct them however; this will be the topic of the next section.

6.2 Construction of Density Functions for Invariant Measures

Recall the family of tent maps with parameters α, h,

$$\phi_{\alpha,h}(x) = \begin{cases} \frac{h}{\alpha}x & \text{if } 0 \leq x < \alpha \\ \frac{h}{1-\alpha}(1-x) & \text{if } \alpha \leq x < 1 \end{cases}$$

Assume α, h are chosen so that the critical point is fixed after a finite number of iterations. If the critical point, (α, h) is fixed at $n+1$ iterations, then denote $A_0, \ldots, A_{n-1}$ the intervals that I, the invariant interval is divided into by the orbit of the critical point. Explicitly, $I = [\frac{h(1-h)}{1-\alpha}, h]$. The orbit of the critical point is then $\{\phi(\alpha), \phi^2(\alpha), \ldots, \phi^{n+1}(\alpha)\}$. Order this set of points, call the ordered set $\{\phi_0(\alpha), \phi_1(\alpha), \ldots, \phi_n(\alpha)\}$. Notice that $\phi_0(\alpha) = \frac{h(1-h)}{1-\alpha}$ and $\phi_n(\alpha) = h$. Then

$$A_0 = [\phi_0(\alpha), \phi_1(\alpha)),\ A_1 = [\phi_1(\alpha), \phi_2(\alpha)),\ \ldots,\ A_{n-1} = [\phi_{n-1}(\alpha), \phi_n(\alpha)].$$

We compute the inverse images of these intervals. From the construction of the intervals, it is not difficult to see that for all i, $\lambda(\phi^{-1}(A_i) \cap A_i) = 0$. Now, based on our numerical experiments (see figures (**??**), (**??**)) our ϕ-invariant measure is assumed to have the form:

$$\mu(A) = \beta_0\, \lambda(A_0 \cap A) + \beta_1\, \lambda(A_1 \cap A) + \ldots + \beta_{n-1}\, \lambda(A_{n-1} \cap A). \tag{3}$$

We set up the following system of equations, let $\bigcup_{i \neq 0} B_i = \phi^{-1}(A_0)$ where we will have $B_i \subset A_i$ for $i \neq 0$. We will need to calculate $\lambda(B_i)$ for each i then $\mu(\phi^{-1}A_0) = \sum_{i \neq 0} \beta_i\, \lambda(B_i)$. From above we know $\mu(A_0) = \beta_0 \lambda(A_0)$. Set $\mu(A_0) = \mu(\phi^{-1}A_0)$ and continue this procedure until the $n-1$ equations are obtained;

$$\mu(A_0) = \mu(\phi^{-1}A_0),\ \mu(A_1) = \mu(\phi^{-1}A_1),\ \ldots,\ \mu(A_{n-1}) = \mu(\phi^{-1}A_{n-1})$$

These equations coupled with $\mu(I) = 1$ give us a system of n equations with n unknowns, we can solve for each β_i uniquely. The β_i's are heights of the steps in the step-function density. We demonstrate this process with a non-trivial example.

6.3 Construction Example, $(\alpha = 1/2, h = 1/\sqrt{2})$

Example 6.7. *Consider the tent map*

$$\phi_{\frac{1}{2}, \frac{1}{\sqrt{2}}}(x) = \begin{cases} \sqrt{2}x & \text{if } 0 \leq x < 1/2 \\ \sqrt{2}(1-x) & \text{if } 1/2 \leq x \leq 1 \end{cases}$$

This is a symmetric tent map with critical value $\phi(\frac{1}{2}) = \frac{1}{\sqrt{2}}$ fixed after 2 iterations. Our invariant interval is $I = [\sqrt{2}-1, \frac{1}{\sqrt{2}}]$ and the critical orbit is $\{\frac{1}{\sqrt{2}}, \sqrt{2}-1, \frac{\sqrt{2}}{\sqrt{2}+1}\}$ and the ordered sequence is $\{\sqrt{2}-1, \frac{\sqrt{2}}{\sqrt{2}+1}, \frac{1}{\sqrt{2}}\}$. We have intervals A_0, A_1 with values,

$$A_0 = \left[\sqrt{2}-1, \frac{\sqrt{2}}{\sqrt{2}+1}\right),\ A_1 = \left[\frac{\sqrt{2}}{\sqrt{2}+1}, \frac{1}{\sqrt{2}}\right]$$

Notice that $\phi^{-1}A_0 = A_1$ and $\phi^{-1}A_1 = A_0$. The system of equations for β_0, β_1 is:

$$\beta_0\left(\frac{\sqrt{2}}{\sqrt{2}+1} - \sqrt{2}+1\right) = \beta_1\left(\frac{1}{\sqrt{2}} - \frac{\sqrt{2}}{\sqrt{2}+1}\right)$$
$$\beta_0\left(\frac{\sqrt{2}}{\sqrt{2}+1} - \sqrt{2}+1\right) + \beta_1\left(\frac{1}{\sqrt{2}} - \frac{\sqrt{2}}{\sqrt{2}+1}\right) = 1$$

The solution to these equations is

$$(\beta_0, \beta_1) = (\frac{3}{2} + \sqrt{2}, 2 + \frac{3}{\sqrt{2}}). \tag{4}$$

Experimental results agree with this solution. See section 7.4 and figure 2.

7 Appendix

These remainder of this report will consist of other work done regarding this project. This includes work I did that is unfinished or work that was unsuccessful in proving existence/uniqueness. The most significant section is section 7.1 which outlines a proof I created to show existence of ergodic measures for a class of tent maps. The first step in the proof is to define the itinerary map and show that it is a homeomorphism from the invariant interval to the sequence space. We then show that the tent maps on the invariant interval are mathematically the same as (conjugate to) the shift operator on the sequence space. We show that ergodicity is preserved in the conjugacy between the shift operator and the tent map. To show ergodicity of the shift operator, we develop a Markov chain argument and show that there is a natural Markov chain associated with the shift operator on the sequence space and that it is ergodic. This shows that our tent maps on the invariant interval (with finite critical orbits) are ergodic and finishes the proof. As a disclaimer to the reader it should be noted that the completeness and accuracy of this proof has yet to be examined thoroughly.
Out of this proof however came another procedure for solving for the density functions of the invariant ergodic measures, when they exist, that is based off of ideas that come from Markov chains. Various other ideas included as well.

7.1 Another Existence Uniqueness Proof

This section gives another proof for the existence and uniqueness of certain ergodic measures developed by myself under the guidance of Dr. Peckham and Dr. James. The validity of this proof as a whole is still under question, so this should be considered an unfinished work.
Consider a tent map $\phi_{\alpha,h}$ (we'll call it ϕ for notational brevity) where the critical point is fixed after a fixed number of iterates, say, l iterates, the orbit of the critical point is then $\{\phi(\alpha), \phi^2(\alpha), \ldots, \phi^l(\alpha)\}$. Add the critical point

to this set and re-label and order the set $\{\phi_0(\alpha), \phi_1(\alpha), \ldots, \phi_l(\alpha)\}$. Let I be the invariant interval for this map. Partition I into the following subintervals: $I_k = [\phi_k(\alpha), \phi_{k+1}(\alpha)]$ for $k = 0, 1, \ldots, l-1$. We have $I = \bigcup_{k=0}^{l-1} I_k$. This is called the Markov partition. The reason that these partitions are interesting to this project is that they provide an association of $\omega \in I$ with elements of Σ_n, the restricted sequence space on the symbols $\{0, 1, \ldots, n-1\}$. The sequence space is restricted because not all configurations are possible or admissible. The following transformation from the invariant interval to the restricted sequence space, Σ_n is called the itinerary map,

Definition 7.1. *The itinerary map, $S : I \to \Sigma_n$ is defined by the following procedure: for all $\omega \in I$ take the sequence $\{\phi^n(\omega)\}_{n=0}^{\infty}$. For each n, $\phi^n\omega \in I_{k_n}$ for some $k_n \in \{0, 1, \ldots, l-1\}$. Set $S(\omega) = (k_0, k_1, \ldots)$.*

Remark 7.2. *In words, if the n^{th} iterate of ω under ϕ is in I_k then the n^{th} element of the symbol sequence for the point $\omega \in I$ is k. S is actually a homeomorphism that determines a conjugacy between ϕ on I and the shift map σ on Σ_n.*

Definition 7.3. *Let $\phi : \Omega \to \Omega$ and $\tau : \Lambda \to \Lambda$ be two functions. ϕ and τ are said to be conjugate if there is a homeomorphism $h : \Omega \to \Lambda$ such that $h \circ \phi = \tau \circ h$.*

We will show that S is a homeomorphism and that the action of the tent maps on I is conjugate to the shift operator on Σ_n under this homeomorphism. This allows the analysis of the tent maps to be reduced to the analysis of the shift operator on the sequence space.

To show S is a homeomorphism, it must be established that S is 1-1 and onto, and S and S^{-1} are both continuous.

Claim 7.4. *S is 1-1.*

Proof. Suppose $x, y \in I$ with $x \neq y$. Assume that $S(x) = S(y)$, this implies that $\phi^n(x)$ and $\phi^n(y)$ both lie in the same interval $I_0, I_1, \ldots, I_{l-1}$. On each of these intervals, ϕ is one-to-one and $|\phi'| = \frac{h}{\alpha}$ or $\frac{h}{1-\alpha} > \mu > 1$ in either case for some $\mu > 1$. So, consider the interval $[x, y]$, for each n, ϕ^n takes this interval in a 1-1 fashion onto $[\phi^n(x), \phi^n(y)]$. The Mean Value Theorem implies that $\lambda([\phi^n(x), \phi^n(y)]) > \mu^n \lambda([x, y])$. Since $\mu^n \to \infty$, there is a contradiction unless $x = y$. □

Remark 7.5. *This result is true only for most points in I. There are a countable collection of points for which this transformation is not well defined. Just as in binary expansions of numbers on the real line, certain points have multiple representations. One solution to this dilemma is to make the intervals $I_0, I_1, \ldots, I_{l-1}$ half open intervals, for example to take $I_0 = [\phi_0(\alpha), \phi_1(\alpha)), \ldots$ This solution gives problems in proving that S is onto. The solution I will adopt is to throw out the set of all points with multiple representations. So,* *when I refer to I from here on in, I will really be referring to $I \setminus B$ where $B = \{\alpha, \phi^{-1}(\alpha), \ldots\} \cup \{\beta_1, \phi^{-1}(\beta_1), \ldots\} \cup \ldots \cup \{\beta_l, \phi^{-1}\beta_l, \ldots\}$, here $\beta_1, \beta_2, \ldots, \beta_l$ are the endpoints of the partition intervals. This will be a set of measure zero in the ϕ-invariant measure μ being constructed due to the fact that $\mu << \lambda$.*

Before S is shown to be onto, I will define admissible sequences in the restricted sequence space. Only certain sequences in the space will have representations in I through S. In the simplest cases with a sequence on more than 3 symbols, a 0 must be followed by a 1 and a 1 must be followed by a 2 and so on. For example, a sequence containing the pattern 102 would not be admissible. The admissible patterns must be taken from the Markov partition.

Claim 7.6. *S is onto.*

Proof. Let $s = s_0 s_1 \ldots$ be an admissible sequence. An $x \in I$ must be found so that $S(x) = s$. Let $I = \bigcup_{m=0}^{l} I_m$ with $l < n$. Also, let I_m be closed intervals for each m. Define

$$\begin{aligned} I_{s_0 s_1 \ldots s_n} &= \{x \in I : x \in I_{s_0}, \phi(x) \in I_{s_1}, \ldots, \phi^n(x) \in I_{s_n}\} \\ &= I_{s_0} \cap \phi^{-1}(I_{s_1}) \cap \ldots \cap \phi^{-n}(I_{s_n}). \end{aligned}$$

We want to show that $I_{s_0 s_1 \cdots s_n}$ form a nested sequence of non-empty closed intervals as $n \to \infty$. First note that $I_{s_0 s_1 \cdots s_n} = I_{s_0} \cap \phi^{-1}(I_{s_1 s_2 \cdots s_n})$. It is clear that for any given s_0, I_{s_0} is closed. Assume that $I_{s_1 s_2 \cdots s_n}$ is closed for induction. We want to know what the inverse image of $I_{s_1 s_2 \cdots s_n}$ is under the tent map. Depending where exactly s_0 is in $\{0, 1, \cdots, l-1\}$, i.e. if $s_0 \in \{0, 1, \cdots, l-5\}$, we can deduce that $s_1 = s_0 + 1$, $s_2 = s_1 + 1, \cdots$. Also, we are looking at behavior as $n \to \infty$ so we can reasonably assume that $n > l$. This gives us that $s_1 s_2 \cdots s_n$ will contain s_{l-1}, so $\phi^{-1}(I_{s_1 s_2 \cdots s_n}) \supset I_{s_0}$ and hence $I_{s_0} \cap \phi^{-1}(I_{s_1 s_2 \cdots s_n})$ will be closed. The other two cases are quite similar, as n gets large, $n > l$ so the argument pushes through in the same way. Now induction will give us that $I_{s_0 s_1 \cdots s_n}$ is a closed interval. These intervals are nested because

$$I_{s_1 s_2 \cdots s_n} = I_{s_1 s_2 \cdots s_{n-1}} \cap \phi^{-n}(I_{s_n}) \subset I_{s_1 s_2 \cdots s_{n-1}}.$$

Therefore, by the Nested Intervals theroem, $\bigcap_{n \geq 0} I_{s_1 s_2 \cdots s_n}$ is non-empty and consists of a single point. This is our $S(x) = s$, so S is onto. Notice that S is onto in the RESTRICTED sequence space only. □

It still requires to be shown that S and S^{-1} are continuous.

Claim 7.7. *S is continuous on Σ_k under the metric on Σ_k defined by:*

Definition 7.8. *Let $x, y \in I$, call $s = S(x) = s_0 s_1 \cdots$ and $t = S(y) = t_0 t_1 \cdots$. Define*

$$d(s,t) = \sum_{n=0}^{\infty} \frac{|s_n - t_n|}{3^n}$$

For a proof that d is a metric, we refer to [5] or [6], the verification is straight forward.

Proof. Again, we are restricting attention only to the restricted sequence space. Take, $x \in I$, say $x \in I_{s_k s_{k+1} \cdots}$ recall that we have thrown out the endpoints of all the intervals $I_{s_k s_{k+1} \cdots}$ in order for S to be well defined for all points in I (all points less the measure 0 kneading sequence). Hence, x is not an endpoint of $I_{s_k s_{k+1} \cdots}$ and thus, if $x \in I_{s_k s_{k+1} \cdots s_n}$ for finite $n > k$, then we can find a $\delta > 0$ we can form a ball of radius δ around x wholly contained in $I_{s_k s_{k+1} \cdots s_n}$. This means that $S(x)$ and $S(y)$ for any $y \in B(x, \delta)$ agree up to the n^{th} term in their expansion. When we look at $d(S(x), S(y))$, we see

$$d(S(x), S(y)) = \sum_{m=0}^{\infty} \frac{|s_m - t_m|}{3^m} = \sum_{m=n+1}^{\infty} \frac{|s_m - t_m|}{3^m} \leq \sum_{m=n+1}^{\infty} \frac{l}{3^m} = \frac{l}{2 \cdot 3^n}$$

Now we see if we choose any $\epsilon > 0$ there is an n such that $\frac{l}{2 \cdot 3^n} < \epsilon$ and for any n we can choose a δ so that for all $y \in B(x, \delta)$ agree up to the n^{th} term for any $x \in I$. Hence for any $\epsilon > 0$ $\exists \delta$ s.t.

$$|x - y| < \delta \Rightarrow d(S(x), S(y)) < \epsilon.$$

So, S is continuous. □

Claim 7.9. *S^{-1} is continuous.*

Proof. To show continuity of S^{-1} we use the same metric d defined above on Σ_k. From the above proof it is clear that for any δ we can pick $s_0 s_1 \cdots$ and $t_0 t_1 \cdots$ such that

$$d(S(x), S(y)) = \sum_{m=0}^{\infty} \frac{|s_m - t_m|}{3^m} < \frac{l}{2 \cdot 3^n} < \delta.$$

We just pick s, t so that the first $n + 1$ terms agree. We also know that given these picked s, t there are $x, y \in I$ with $S(x) = s$ and $S(y) = t$, furthermore, if $S(x) = s_0 s_1 \cdots$ and $S(y) = t_0 t_1 \cdots$ then if $x \in I_{s_k s_{k+1} \cdots s_n}$ then $y \in I_{s_k s_{k+1} \cdots s_n}$. We also know that neither x nor y can be an endpoint of $I_{s_k s_{k+1}, \cdots s_n}$. So, we set $\gamma = length(I_{s_k s_{k+1}, \cdots, s_n})$ and if we fix x then for any $\epsilon > 0$, there exists an $\gamma < \epsilon$ such that if we choose $\delta < \frac{l}{2 \cdot 3^n}$ then $d(S(x), S(y)) < \delta \Rightarrow |x - y| < \epsilon$. □

This shows that S is a homeomorphism.

Remark 7.10. *It is automatic that since S is a homeomorphism between I and the restricted sequence space that ϕ on I is conjugate to σ on the restricted sequence space Σ_n via this homeomorphism S.*

The point and purpose of showing that S is a homeomorphism is that conjugacies preserve ergodicity. At times, it is easier to work with transformations on the sequence space rather than the interval. This conjugacy has been developed to work with the shift operator on the restricted sequence space.

Claim 7.11. *Conjugacies preserve ergodicity. If ϕ is an ergodic transformation, and ϕ is conjugate to τ, then τ is ergodic with respect to an induced measure.*

Proof. To show ergodicity it must be shown that

1. τ is measure preserving: Let ϕ be an ergodic transformation acting on the probability space $(\Omega, \mathcal{F}, \mu)$ and τ a transform acting on $(\Sigma, \mathcal{L}, \nu)$. Let $h : (\Omega, \mathcal{F}) \to (\Sigma, \mathcal{L})$ be a homeomorphism with the property that $h \circ \phi = \tau \circ h$, also, say h has inverse function g, then $\phi^{-1} \circ g = g \circ \tau^{-1}$.
   Since ϕ is ergodic then $\mu(\phi^{-1}A) = \mu(A)$, if $A = g(B)$ then $\mu(\phi^{-1} \circ g(B)) = \mu(g(B))$ where $A \in \mathcal{F}$ and $B \in \mathcal{L}$. From the conjugacy relationship $\mu(g(B)) = \mu(g \circ \tau^{-1}B)$. We can then write,
   $$(\mu \circ g)(B) = (\mu \circ g)(\tau^{-1}B)$$
   so it is clear that τ preserves $\mu \circ g$ and $\mu \circ g$ is a probability measure on $(\Sigma, \mathcal{L})$ (this is the afore-mentioned induced measure).

2. the σ-algebra of invariant events for τ is trivial: Let $\mathcal{I}$ be the σ-algebra generated by the invariant events for ϕ. If $A \in \mathcal{I}$ then $\phi^{-1}A = A$ so, $\phi^{-1}(g(B)) = g(B)$ for g, B as in the proof of (1). Invoking the conjugacy identity we see $g(B) = g \circ \tau^{-1}(B)$. This implies that if A is invariant for ϕ, then $B = h(A)$ is invariant for τ. It is easy to see that this relationship will be preserved in reverse as well. This gives us a complete characterization for the σ-algebra of invariant events for τ, denoted $\mathcal{J}$. What remains to be shown is that $\nu \equiv \mu \circ g << \mu$.
   Assume that $A \in \mathcal{F}$ with $\mu(A) = 0$. Since g is a homeomorphism, there is a $B \subset \Sigma$ with $h(A) = B$. Notice that $\mu(A) = \mu(g \circ (h(A)) = (\mu \circ g)(B) = 0$. So if A has measure zero, then its image under g in the sequence space has measure 0 in the induced measure ν. Hence $\nu << \mu$ and τ is ergodic.

□

7.1.1 Proof that σ is ergodic on Σ_n

We now state a few definitions and theorems that will be useful in the coming proofs,

Definition 7.12. *We define the probability measure P_π on the sequence space using Kolmogorov's theorem. Let p_n be a sequence of transition probabilities and μ an initial distribution on $(S, \mathcal{L})$, we can define a consistent set of finite-dimensional distributions by*

$$P(X_j \in B_j,\, 0 \le j \le n) = \int_{B_0} \mu(dx_0) \int_{B_1} p_1(x_0, dx_1) \cdots \int_{B_n} p_n(x_{n-1}, dx_n)$$

Then Kolmogorov's theorem allows us to construct a probability measure P_μ on the sequence space $(S^{\{0,1,\cdots\}}, \mathcal{L}^{\{0,1,\cdots\}})$ so that coordinate maps $X_n(\omega) = \omega_n$ have the desired distributions. See [1] p. 239 for a full discussion.

Theorem 7.13. *(Lévy's 0-1 Law) If $A \in \sigma(\mathcal{F}_n,\, n \geq 1) \supset \mathcal{T}$, then $E(\mathbf{1}_A|\mathcal{F}_n) \to \mathbf{1}_A$ a.s.*

Theorem 7.14. *(Markov Property) Suppose that X_n are the coordinate maps on the sequence space, $\mathcal{F}_n = \sigma(X_0, X_1, \cdot, X_n)$ and for each initial distribution we a measure P_μ defined in (**??**) that makes X_n a Markov chain. Ω_0 is the sequence space on the symbols $\{0, 1, \cdots\}$. Let $Y : \Omega_0 \to \mathbb{R}$ be bounded and measurable. $E_\mu(Y \circ \sigma^n|\mathcal{F}_n) = E_{X(n)}Y$ where the subscript μ indicates that the conditional expectation is taken with respect to P_μ. $X(n)$ is a function $X(n) = x$.*

Remark 7.15. *It should be noted that the proof of theorem (**??**) applies only to certain cases of maps with a finite critical orbit, not all.*

That being established, we turn back to our Markov partitions. A few properties of this partition must now be proved,

1. There is a natural Markov chain on the symbols $0, 1, \cdots n-1$ associated with this partition.

   Proof. Take the function $X : \Sigma_n \to \{0, 1, 2, \cdots n-1\}$ given by,

   Definition 7.16. *for $s \in \Sigma_n$, say $s = s_0 s_1 \cdots$ then $X(s) = s_0$. If σ is the shift operator then the mentioned Markov chain is $\{X(s), X(\sigma s), X(\sigma^2 s), \cdots\}$. For notational convenience we say $X_n(s) = X(\sigma^n(s))$.*

   In order to show this is a Markov chain, it must be shown that

   $$P(X_n(s) = k_n|X_{n-1}(s) = k_{n-1}, \cdots, X_0(k_0)) = P(X_n(s) = k_n|X_{n-1}(s) = k_{n-1})$$

   for any n and any k. Assume the critical point is fixed after l iterations, let $I = \bigcup_{k=0}^{l-1} I_k$ be the Markov partition. Notice first that for all $\omega \in I_0$ that $\phi(\omega) \in I_1$ and also, for $\omega \in I_k$, $\phi(\omega) \in I_{k+1}$ for all $k = 0, 1, \cdots l-5$. If $\omega \in I_{l-4}$ then $\phi(\omega) \in I_{l-3}$ or I_{l-2} with probabilities p, q respectively. For $\omega \in I_{l-3}$ and $\omega \in I_{l-2}$ then $\phi(\omega) \in I_{l-1}$. For all $\omega \in I_{l-1}$, $\phi(\omega) \in I_k$ for $k = 0, 1, \cdots, l-2$ with probabilities $p_0, p_1, \cdots, p_{l-2}$ respectively. We will have $\Sigma_{k=0}^{l-2} p_k = 1$. We can represent this relationship with the following flow map:

   $$I_0 \overset{prob.1}{\rightarrow} I_1 \overset{prob.1}{\rightarrow} \cdots \rightarrow I_{l-4} \begin{array}{l} \overset{p}{\rightarrow} I_{l-3} \\ \overset{q}{\rightarrow} I_{l-2} \end{array} \overset{prob.1}{\rightarrow} I_{l-1} \begin{array}{l} \overset{p_0}{\rightarrow} I_0 \\ \overset{p_1}{\rightarrow} I_1 \\ \quad\vdots \\ \overset{p_{l-2}}{\rightarrow} I_{l-2} \end{array}$$

   This means that if $X_0(s) = 0$ then $X_1(s) = 1$ and if $X_i(s) = k$, then $X_{i+1}(s) = k+1$ for any $i \in \{0, 1, \cdots, l-5\}$. Intuitively this is a Markov chain because the images of each I_j under ϕ are $\bigcup_{k\in N} I_k$ for some finite index set N not containing j. The images of I_j under ϕ are being distributed uniformly because ϕ is a linear function. There will also be only one pre-image for any given itinerary. This is due to the nature of the Markov partition which makes the map 1-1 on each partition. The transition probabilities are proportional to the lengths (the Lebesgue measure) of the images of I_j. Now we calculate the probabilities in the above flow chart.

$$p = \frac{\lambda(I_{l-3})}{\lambda(I_{l-3} \cup I_{l-2})} \text{ and } q = \frac{\lambda(I_{l-2})}{\lambda(I_{l-3} \cup I_{l-2})}$$

and

$$p_i = \frac{\lambda(I_i)}{\lambda(\bigcup_{n=0}^{l-2} I_n)} \text{ for } i = 0, 1, \cdots l-2.$$

These probabilities will be the same no matter what the itinerary of the sequence, since the images will always be uniformly distributed, and hence the transition probabilities will be proportional to the lengths of the images of the present interval.

□

Also, this Markov chain is nice because most of these probabilities are certain, many transitions occur with probability 1. For example, let $k_n k_{n-1} \cdots k_0$ be an admissible sequence, if $k_i \in \{0, 1, \cdots, l-5\}$ then it is followed by $k_i + 1$ with probability 1. If $k_i = l-1$ then it is followed by $\{0, 1, \cdots l-2\}$ with the above stated probability.
The Markov chain has the following properties,

(a) It is irreducible. This should be clear from the above diagram, the diagram has the intervals in the Markov partition, the intervals correspond directly to the transitions between states. Each state being positive recurrent follows because we have a finite state Markov chain with one communicating class.

(b) The chain is aperiodic: $l-1 \to l-2 \to l-1$ and $l-1 \to l-4 \to l-2 \to l-1$, so $gcd\{2,3\} = 1 \Rightarrow d_{l-1} = 1$ and 1 communicating class implies the period of the chain is 1 for any $l > 3$. In the case where $l = 3$ the chain has period 2 (shown in a later section by example) and the cases $l = 2$ and $l = 1$ are not interesting.

2. The Markov chain is ergodic. From above, it should be clear that a stationary distribution π for X_n exists and $\pi(x) > 0$, for $x \in \{0, 1, \cdots, l-1\}$. We want to show that the σ-algebra generated by the class of invariant events $\mathcal{J}$ is trivial.
Let σ^n be the n-shift operator, if $A \in \mathcal{J}$ then $\mathbf{1}_A \circ \sigma^n = \mathbf{1}_A$. This is easy to see; let X be a $\mathcal{J}$-measurable function, then $\{\omega : X(\omega) \in B\} \in \mathcal{J}$ for any Borel set B. This means that for Borel set B, $X^{-1}(B)$ is invariant, so $\phi^{-1}(X^{-1}B) = X^{-1}B$. This can be restated as $\{\omega : (X \circ \phi)(\omega) \in B\} = \{\omega : X(\omega) \in B\}$. Now pick the very particular Borel set $B = \{x\}$. We get $\{\omega : (X \circ \phi)(\omega) = x\} = \{\omega : X(\omega) = x\}$, the result follows.
Let $\mathcal{F}_n = \sigma(X_0, X_1, \cdots, X_n)$, the shift-invariance of $\mathbf{1}_A$ and **(??)** imply

$E_\pi(\mathbf{1}_A|\mathcal{F}_n) = E_\pi(\mathbf{1}_A \circ \sigma^n|\mathcal{F}_n) = h(X_n)$ where $h(x) = E_x\mathbf{1}_A$. Here, E_π is the expectation with respect to the probability measure P_π defined by (**??**) with $\mu = \pi$. $E_x\mathbf{1}_A$ represents taking the expected value of $\mathbf{1}_A$ with respect to the random variable $X_n = x$. (**??**) implies $E_\pi(\mathbf{1}_A|\mathcal{F}_n) \to \mathbf{1}_A$ a.s. and since X_n is irreducible and positive recurrent, then for any $y \in \{0, 1, \cdots, l-1\}$, $E_{X_n}\mathbf{1}_A = E_y\mathbf{1}_A$ i.o., so either $h(y) = 0$ or 1 which implies $P_\pi(A) \in \{0, 1\}$. Hence the Markov chain $\{X_n\}_{n=0}^\infty$ is ergodic. So, $\{X_n\}_{n=0}^\infty = \{X(\sigma^n s)\}_{n=0}^\infty$ is ergodic.

Corollary 7.17. *ϕ is ergodic on I.*

Proof. It was established in (**??**) that the shift map σ on Σ_n is conjugate to ϕ on I and conjugacy preserves ergodicity. Hence the ergodicity of σ on Σ_n implies the ergodicity of ϕ on I provided that the orbit of the critical value is finite. □

This not only concludes the second proof of this fact, but it is more general; it holds not only for the full height symmetric map, but any map with a finite critical orbit. The proof is given more for landing on a repelling fixed point, but it appears to generalize easily to the repelling periodic point case.

7.2 An Example of Solving for the Invariant Densities using Markov Chains

The Markov chain proof of ergodicity provides a constructive method for solving for the densities of the invariant measure. For this example we take our tent function to be the same as in the previous constructive example, let ϕ be defined as in (**??**). First, we need to know the transition probabilities. The Markov partition consists of 3 intervals:

$$I_0 = \left[\sqrt{2}-1, \frac{1}{2}\right], \; I_1 = \left[\frac{1}{2}, \frac{\sqrt{2}}{\sqrt{2}+1}\right], \; I_2 = \left[\frac{\sqrt{2}}{\sqrt{2}+1}, \frac{1}{\sqrt{2}}\right]$$

We have the flow map

$$\begin{array}{ccc} I_0 \overset{prob.1}{\rightarrow} & & \overset{p}{\rightarrow} I_0 \\ & I_2 & \\ I_1 \overset{prob.1}{\rightarrow} & & \overset{q}{\rightarrow} I_1 \end{array}$$

where

$$p = \frac{\lambda(I_0)}{\lambda(I_0) + \lambda(I_1)} = \frac{\frac{3}{2} - \sqrt{2}}{3 - 2\sqrt{2}} = \frac{1}{2} = q$$

Then we have for the transition probability matrix:

$$\begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ 1/2 & 1/2 & 0 \end{pmatrix}$$

We solve $\pi P = \pi$ where $\pi = (\pi_0 \; \pi_1 \; \pi_2)$ gives $\pi_0 = \pi_1 = 1/4$ and $\pi_2 = 1/2$. To find the heights of the step functions in the distribution for the invariant measure, normalize these probabilities with the length of their corresponding the intervals:

$$\beta_0 = \frac{\pi_0}{\lambda(I_0)}\,,\ \beta_1 = \frac{\pi_1}{\lambda(I_1)}\,,\ \beta_2 = \frac{\pi_2}{\lambda(I_2)}.$$

Computing the values gives:

$$\beta_0 = \beta_1 = \frac{1/4}{3/2 - \sqrt{2}} = \frac{3}{2} + \sqrt{2}\,,\ \beta_2 = \frac{1/2}{\frac{\sqrt{2}-1}{2+\sqrt{2}}} = 2 + \frac{3}{2}\sqrt{2}$$

This agrees with the results of the first example, (**??**).

7.3 Experimental Results

My initial stages in this project consisted of numerical simulation and experimentation. To gain insight into the form of the invariant measures for finite critical orbit cases I tried iteration of random (or special) points in the invariant interval and observing the behavior under iteration. From these experiments it was conjectured that the form of the invariant measures were step functions.
A word on the experimental results:
The first simulations were run by iterating random points in a map where the critical point was fixed after 2 iterations. The result of this can be seen in figure (**??**), suggesting a probability density which is constant on 2 subintervals. Another example was a map with the critical orbit fixed at 3 iterations, this result can be seen in figure (**??**).

Experimentation was also done under the assumption that smooth (C^1) functions were needed to apply the Misiurewicz result. Since the tent maps are not C^1-smooth, a smooth approximation was created. This assumption is actually not needed since the Misiurewicz result applies directly to the tent maps. The smoothed tent maps are formed as follows:

$$\phi_{\alpha,h,\epsilon}(x) = \begin{cases} \frac{h}{\alpha}x & 0 \le x < \alpha - \epsilon \\ a + bx + cx^2 + dx^3 + ex^4 + kx^5 & \alpha - \epsilon \le x \le \alpha + \epsilon \\ \frac{h}{1-\alpha}(1-x) & \alpha + \epsilon < x \le 1 \end{cases}$$

The coefficients a, b, c, d, e, k can be uniquely solved for in terms of α, ϵ (I solved for them using Mathematica). It can be shown that these approximations satisfy all conditions of the Misiurewicz theorem, the only condition that is non-trivial is condition 3, that the Schwarzian derivative for the smoothed functions be non-positive. This can be shown by direct computation once the coefficients are computed.
For visualization I created several histograms corresponding to special cases of tent maps where the critical point $1/2$ is fixed after 2 iterations in the first set, and 3 iterations in the second set. Each set has 4 figures. The first is the density for the tent map with the above parameter values, the next 3 figures are a progression of histograms for the smoothing functions associated with the particular tent map with epsilon = .1, .01 and .001. These show the convergence of the smoothing functions' density to the step function of the tent map.

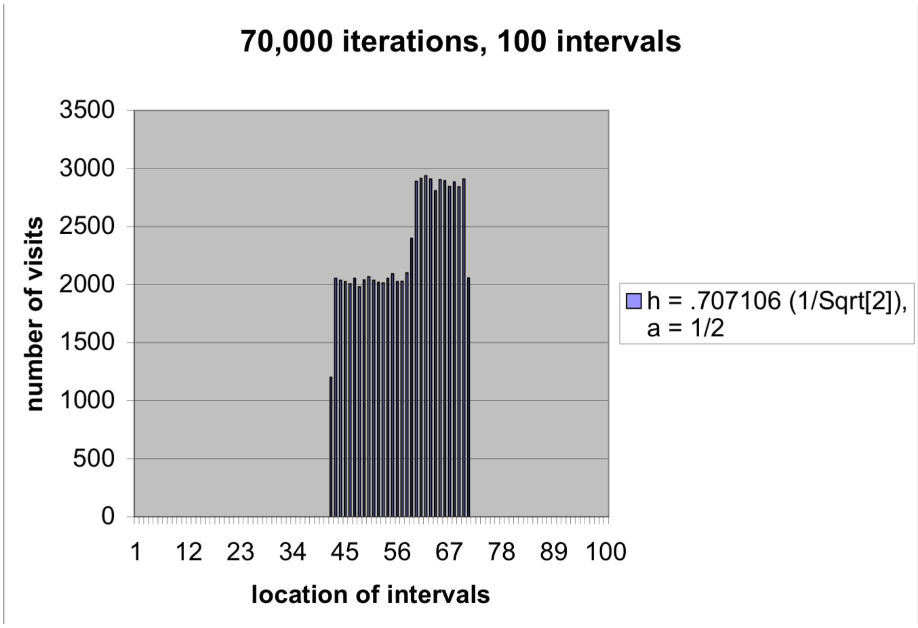

Figure 2: Critical orbit fixed after 2 iterations

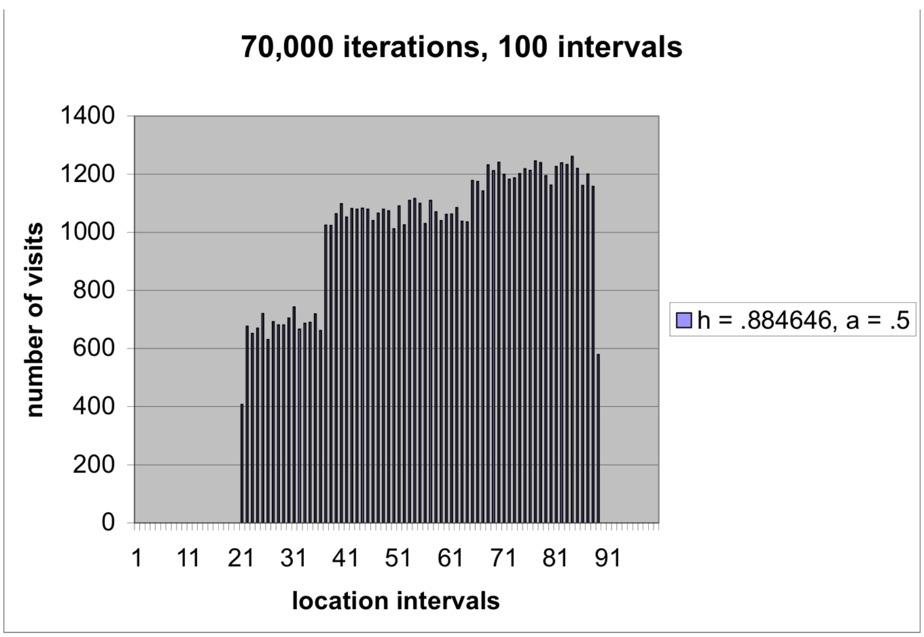

Figure 3: Critical orbit fixed after 3 iterations

A word on the titles for the graphs: most are labeled by the number of step functions in the density. 70,000 or 70k represents the number of iterations of the random point, 100 or 1k represents the number of subintervals the unit interval was divided into for the histogram.

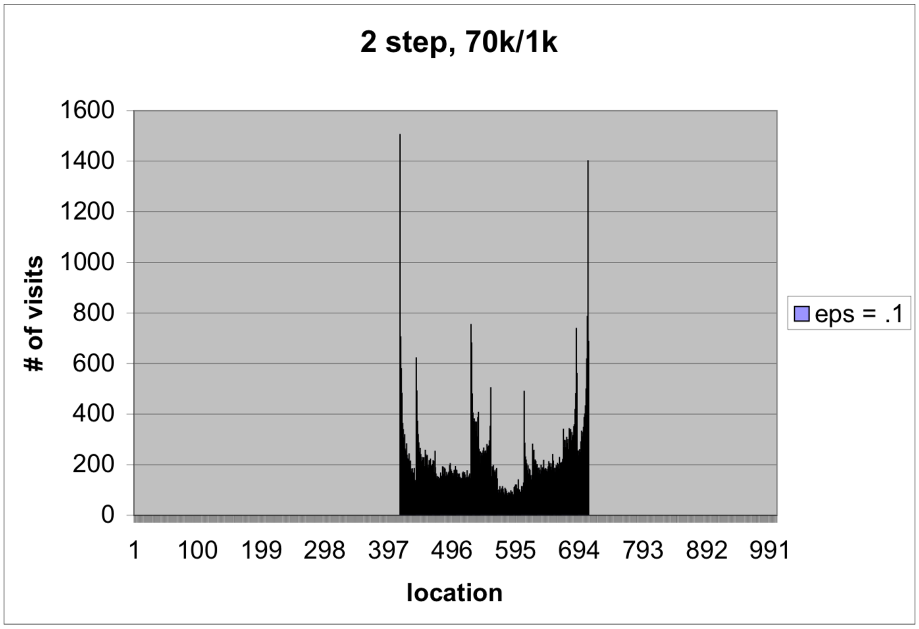

Figure 4: critical orbit fixed after 2 iterations

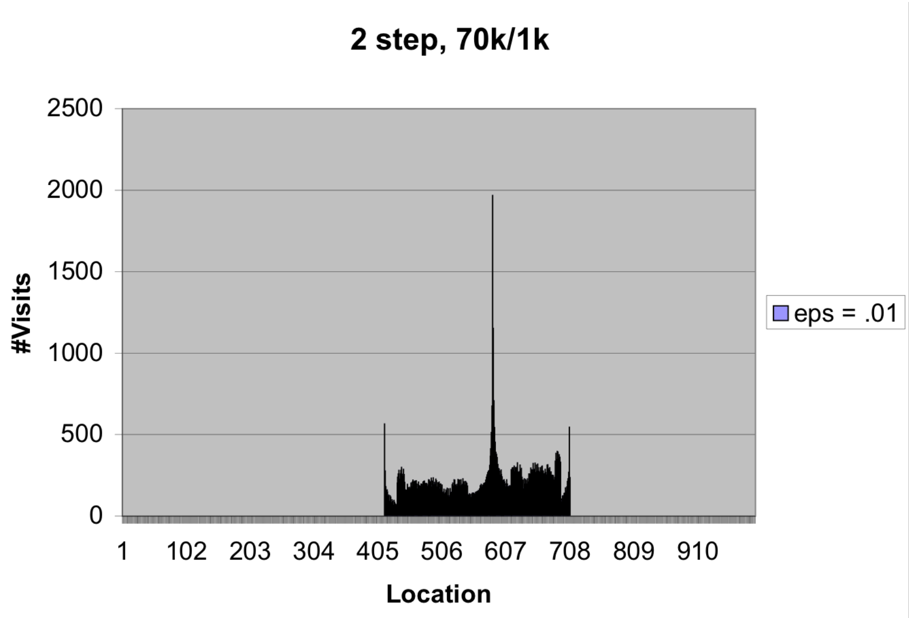

Figure 5: critical orbit fixed after 2 iterations

8 References

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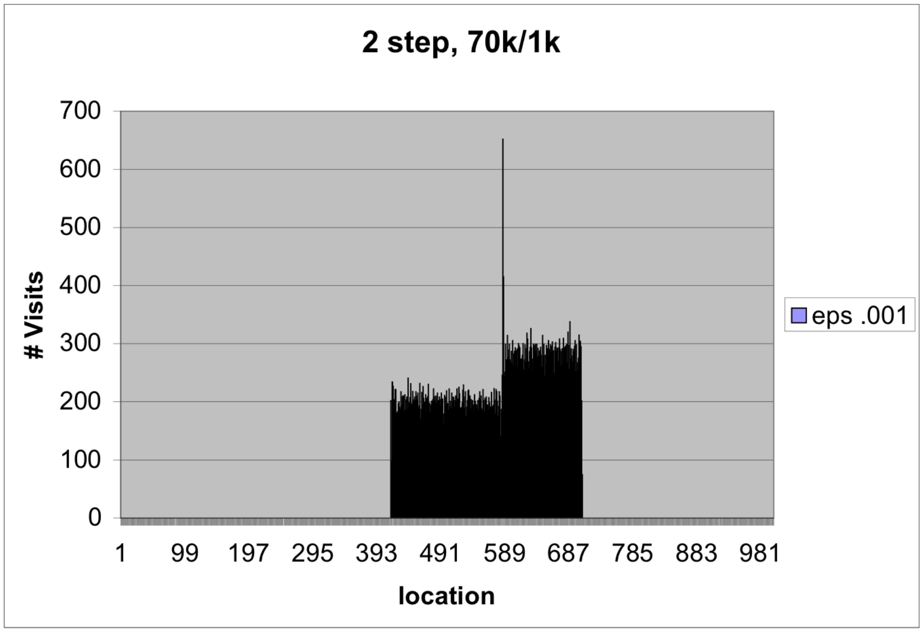

Figure 6: critical orbit fixed after 2 iterations

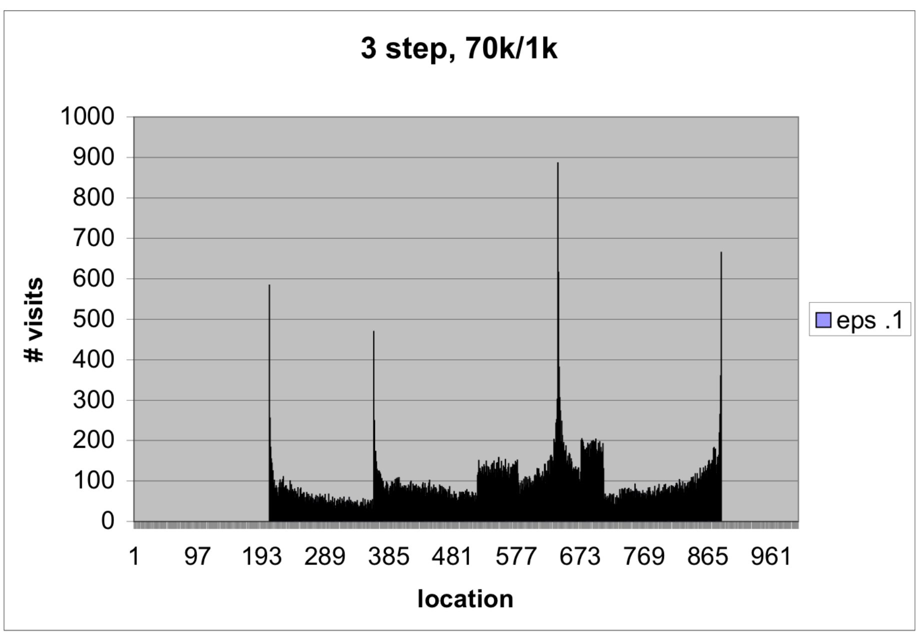

Figure 7: critical orbit fixed after 3 iterations

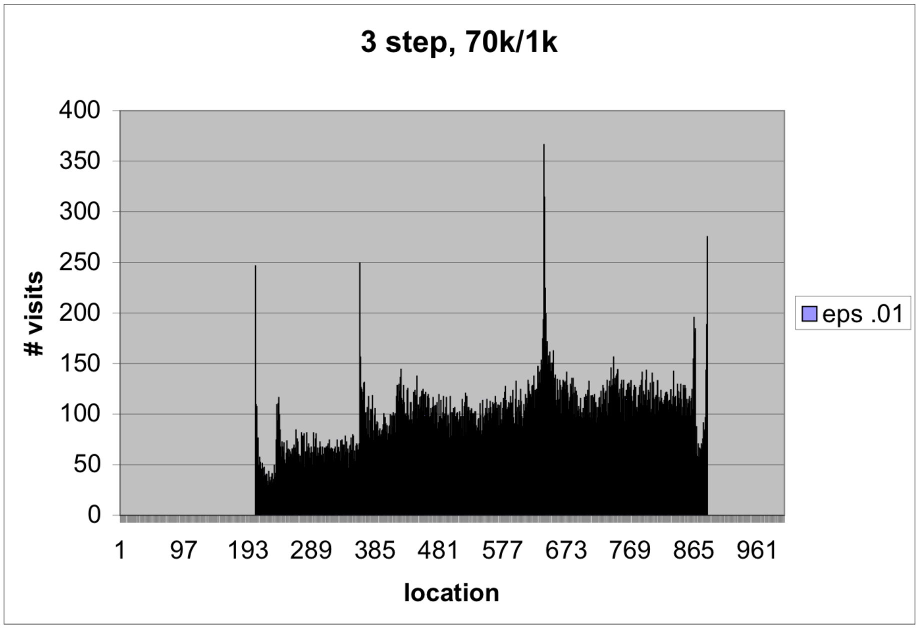

Figure 8: critical orbit fixed after 3 iterations

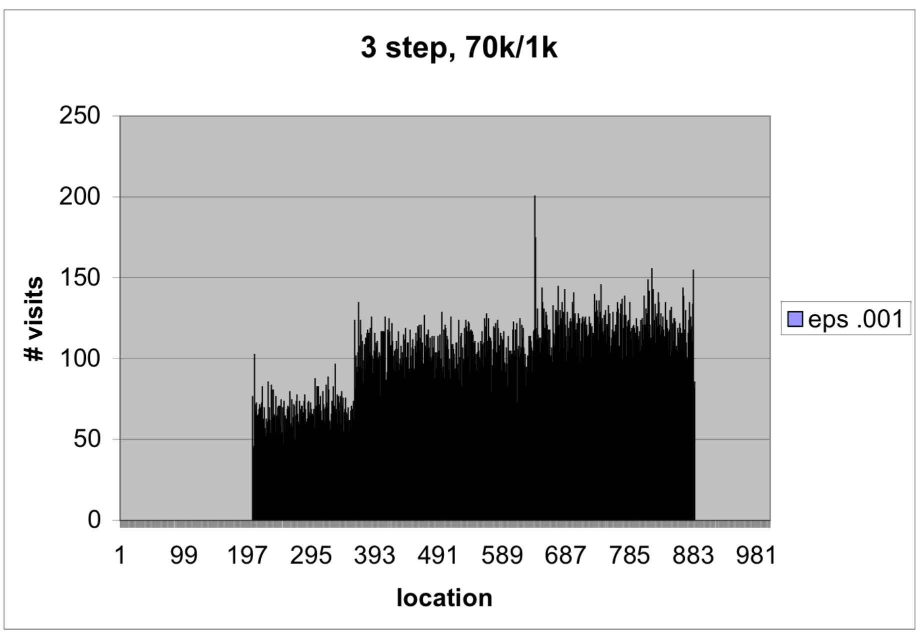

Figure 9: critical orbit fixed after 3 iterations