

**The Spine of Newton's Method for Cubics:  
Dynamics for Isosceles Root Configurations**

by

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Dynamics for Isosceles Root Configurations**

A PLAN B PAPER

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BY

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**Abstract:**

Solving an equation of the form  $f(z) = 0$  to find the complex roots of the function  $f$  is a common problem in applied mathematics. A common iterative numerical method for finding roots is Newton's Method. Newton's method is commonly known to fail to converge to a root if the "starting point" is close to equidistant from a pair of roots of the function. This report/paper is an extension of a paper titled "Newton's Versus Halley's Method: An Approach Via Complex Dynamics" by Roberts and Horgan-Kobelski[2004]. Using a reduction of parameters and analytic techniques, the authors proved the existence of superattracting period- $n$  orbits for all  $n$  under Newton's Method. Their proof involved looking at a single real parameter family of real rational maps which had implications for the full complex Newton's Method, especially the complex cubic polynomials where the roots form an isosceles triangle. This paper extends their results by analyzing more completely the real rational family. Besides the superattracting period- $n$  orbits found by the authors, numerical evidence shows that many other superattracting periodic orbits are born as the parameter is varied. Classifying these other superattracting periodic orbits by rotation numbers is done by using a conjugate circle map.

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I would also like to my family. I would like to thank them for their unwavering support throughout my educational career.

**1. Introduction:** Solving an equation of the form  $f(z) = 0$  to find the complex roots of the function  $f$  is a common problem in applied mathematics. A common iterative numerical method for finding roots is Newton's Method, discovered by Sir Isaac Newton, and published in *The Method of Fluxions* in 1736. Although Newton's method has been used to find roots of functions for many years, it is commonly known to fail to converge to a root if the "starting point", "initial condition", or "seed" is close to equidistant from a pair of roots of the function. This report/paper is an extension of a paper titled "Newton's Versus Halley's Method: An Approach Via Complex Dynamics" by Dr. Gareth E Roberts and Jeremy Horgan-Kobelski[2004]. Their paper compared and contrasted the numerical root finding differences between Newton's and Halley's Method for complex cubic polynomials. Using a reduction of parameters and analytic techniques, Roberts and Horgan-Kobelski proved the existence of superattracting period- $n$  orbits for all  $n$  under Newton's Method and Halley's Method. This research paper deals only with Newton's Method for cubics and extends the Roberts and Horgan-Kobelski results. More specifically, we examine complex cubic polynomials where the roots form an isosceles triangle. Besides the superattracting period- $n$  orbits found by Roberts and Horgan-Kobelski, numerical evidence shows that many other superattracting periodic orbits exist just on the perpendicular bisector of two roots for varying values of the parameter. Classifying these other superattracting periodic orbits by rotation numbers is done by using a conjugate circle map.

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presents some of the previous results from the Roberts and Horgan-Kobelski paper along with some basic motivation for the topic of this paper. Chapter 4 introduces numerical evidence and graphical evidence of periodic orbits. Chapter 4 also looks at the basin of attraction for the fixed point and periodic orbit both for a specific parameter value along with a general phase space. Chapter 5 introduces circle maps as a tool to classify the periodic orbits found in chapter 4. Chapter 6 provides some conjectures based on graphical evidence from the circle maps. Chapter 7 contains the code and tables used or developed in this project.

## 2. Preliminaries:

Newton's Method for finding roots of a complex polynomial  $p$  is iterating the rational map:

$$N_p(z) = z - \frac{p(z)}{p'(z)}$$

Newton's Method has been studied for many years and much is known for linear, quadratic, and cubic functions. Cubic functions, although studied, still have a lot to be discovered.

It is known that if  $p$  is degree  $d$  and has distinct roots (complex or real), then  $N_p$  is a rational map of degree  $d$ . It is clear that the simple roots  $\alpha_n$  of  $p(z)$  are the fixed points of  $N_p$  as shown by a simple computation:

$$N_p(\alpha_n) = \alpha_n - \frac{p(\alpha_n)}{p'(\alpha_n)} = \alpha_n - \frac{0}{p'(\alpha_n)} = \alpha_n, \rightarrow N_p(\alpha_n) = \alpha_n$$

providing that  $p'(\alpha_n) \neq 0$ .

Again, a short computation shows that the roots of  $p$  are superattracting (the derivative on  $N$  at the fixed point is zero):

$$N'_p(z) = \frac{p(z)p''(z)}{[p'(z)]^2}$$

$$N'_p(\alpha_n) = \frac{p(\alpha_n)p''(\alpha_n)}{[p'(\alpha_n)]^2} = \frac{0 \cdot p''(\alpha_n)}{[p'(\alpha_n)]^2} = 0$$

It is also important to note that any inflection points,  $r_i$ , of  $p$  are also critical points of  $N_p$ ; they are called *free* critical points. The term *free* critical point comes from the fact that the fate of a free critical point is not always known, whereas the roots of  $p$  are always the fixed points of  $N_p$ .

Critical points are heavily studied in complex dynamics because it is known that if there exists a superattracting periodic orbit, a critical point must be on the orbit. This comes from the idea that a superattracting periodic orbit has an iterate that lands on a point where the derivative is zero. Since the critical points are the only points that have the derivative being zero, at least one critical point must lie on the periodic orbit. Moreover, critical point orbits are studied because not only do they find all superattracting period- $n$  orbits, but they find all attracting period- $n$  orbits. A big theorem using the Schwarzian Derivative states that if there exists an attracting period- $n$  orbit for a fixed parameter value, then the orbit of some critical point will be attracted to that period- $n$  orbit.

Attracting periodic orbits are essentially “bad” for Newton’s method because they imply that there exist whole regions of initial starting points for which Newton’s Method

fails. While this is bad for Newton and his algorithm, it provides us with some very interesting dynamics.

Finding the roots of a linear function with Newton's Method is trivial, as long as the linear function is not constant. Newton's Method converges to the unique root after one iteration.

Newton's Method for quadratic functions is much more complicated. A quadratic function has two roots  $x_{\pm}$ . There exists a perpendicular bisector between  $x_{\pm}$  that divides where orbits will converge. Any orbit with initial seed closer to the root  $x_{-}$  will necessarily converge to  $x_{-}$ , and similarly any orbit initial seed closer to the root  $x_{+}$  will necessarily converge to  $x_{+}$ . On the perpendicular bisector however, Newton's Method fails to locate a root of the quadratic function. The bisector is invariant and is known to exhibit chaotic behavior.

Newton's Method for complex cubic functions, although heavily studied, is very complicated and still has many unsolved areas [J.H Curry, L. Garnett and D. Sullivan, 1983]. One of the reasons for the difficulty of the cubic family is that it has four complex parameters. Newton's Method for complex cubic polynomials has 4 critical points, as can be seen by the numerator of  $N'_p(z)$  on page 6. Three critical points are classified as just critical points and are the roots of the polynomial (fixed points of  $N_p$ ), while the fourth critical point is called a "free" critical point and is found by finding the inflection point of the cubic polynomial.

## 2.1. Basic Definitions

The following definitions are taken from Devaney[1992].

**Orbit:** Given an  $z_0$ , we define the orbit of  $z_0$  under a function  $F$  to be the sequence of points  $z_0, z_1 = F(z_0), z_2 = F^2(z_0), \dots, z_n = F^n(z_0)$ . A *periodic* orbit is a sequence of points which eventually repeats itself.

**Fixed Point:** A fixed point is a point  $z_0$  that satisfies  $F(z_0) = z_0$ . The orbit of the fixed point is constant:  $z_0, z_0, z_0, z_0 \dots$

**Periodic Point:** A periodic point is a point which lies on a periodic orbit.

**Attracting Point:** A point  $z_0$  is attracting if  $|F'(z_0)| < 1$

**Superattracting Point :** A point  $z_0$  is superattracting if  $|F'(z_0)| = 0$

**Repelling Point:** A point  $z_0$  is repelling if  $|F'(z_0)| > 1$

**Attracting Periodic Orbit:** Given an orbit :

$z_0, z_1 = F(z_0), z_2 = F^2(z_0), \dots, z_n = F^n(z_0)$ , the orbit is attracting if

$$|F'(z_0)| \cdot |F'(z_1)| \cdot \dots \cdot |F'(z_n)| < 1$$

**Superattracting Periodic Orbit:** Given an orbit :

$z_0, z_1 = F(z_0), z_2 = F^2(z_0), \dots, z_n = F^n(z_0)$ , the orbit is superattracting if

$$|F'(z_0)| \cdot |F'(z_1)| \cdot \dots \cdot |F'(z_n)| = 0$$

**Repelling Periodic Orbit:** Given an orbit :

$z_0, z_1 = F(z_0), z_2 = F^2(z_0), \dots, z_n = F^n(z_0)$ , the orbit is attracting if

$$|F'(z_0)| \cdot |F'(z_1)| \cdot \dots \cdot |F'(z_n)| > 1$$

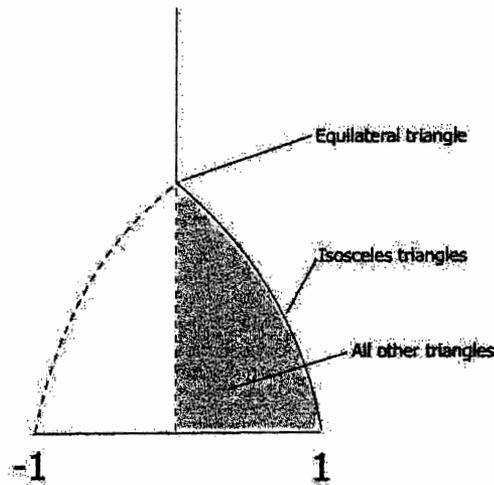
### 3. Background

#### 3.1. Reduction from a 4 parameter family to a 1 parameter family:

Because of the difficulty of the complex cubic family under Newton's Method a reduction from a four parameter family to a one parameter family is constructed as follows[Blanchard,1991, Roberts and Horgan-Kobelski,2004]:

Given any triangle formed by the three distinct roots  $\alpha_1, \alpha_2, \alpha_3$  in the complex plane there is a similarity transformation that takes the longest side of the triangle to the real interval  $[-1,1]$  and the remaining vertex to a point in the upper half plane above the segment of the real axis  $[-1,1]$ . A *similarity transformation* is a map  $T : \mathbb{R}^k \rightarrow \mathbb{R}^k$  that is the composition rotations, flips, and translations. By making use of a similarity transformation, we can restrict the study of almost all cubic polynomials to the one parameter family  $p_\lambda(z) = (z-1)(z+1)(z-\lambda)$ ,  $\lambda \in \mathbb{C}$  and  $z \in \mathbb{C}$ . The parameter space is depicted by the following diagram. The parameter space is geometrically constructed by drawing 2 circles with radius 2 around the points 1, and -1. The intersection of these circles in the upper half plane is on the perpendicular bisector of the line between -1 and 1. Usually the parameter value  $\lambda$  is chosen anywhere on the right arc of the "triangle" or in the interior (gray area) of the triangle. If a parameter value  $\lambda$  is chosen outside of this region, another similarity transformation would bring the parameter value into the triangular region. The vertical line passing through the triangle is referred to as the spine and is the imaginary axis in the complex plane. Parameter values on the right (or left) arc correspond to isosceles triangles. Any parameter value  $\lambda$  chosen on the left arc is equivalent to choosing a value on the right arc, hence the dotted line. The parameter values chosen on the left and right arcs are also equivalent to choosing a parameter value

on the solid part of the spine. This can be visualized by starting with the upper vertex of the isosceles triangle and rotating the triangle so the vertex is on the imaginary axis. Then take the base of the isosceles triangle formed by the roots and scale the triangle so that the base of the triangle fits to the region  $[-1,1]$ , again using a similarity map. The only cubic polynomial that cannot make use of this similarity conjugacy is the cubic polynomial  $z \mapsto z^3$  (1 root with multiplicity 3). The family is chosen for the symmetry that it possesses about the imaginary axis, both in the parameter plane and in the dynamic plane.



Parameter Space for Complex Cubic Polynomials

Note: Roberts and Horgan-Kobelski's parameter space triangle was shaded in completely because they used an affine transformation. However, using a similarity transformation which is 1:1 for parameter values in the right half of the triangle to the equivalent classes of complex cubics, we can reduce the parameter space to include only the right side of parameter space triangle and the "dotted" spine in the interior of the triangle.

### 3.2. Critical Point Behavior

In our case we only have 4 critical points: 3 fixed and 1 free critical point. Since the fate of 3 of the critical points (the roots of the cubic polynomial) is known, all attracting periodic orbits can be found by iterating the free critical point. The free critical point  $r = \lambda/3$  is verified by the following short computation:

$$p_\lambda(z) = (z-1)(z+1)(z-\lambda)$$

$$p_\lambda''(z) = 4z + 2(z-\lambda)$$

$$p''(z) = 0 \Rightarrow z = \frac{\lambda}{3}$$

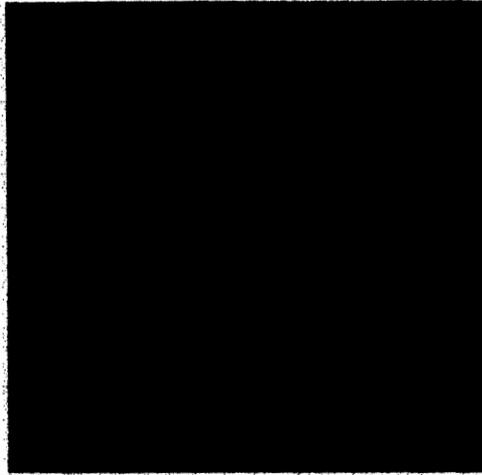


Mandelbort-like Picture on the Parameter Space

The Mandelbrot-like picture made by Roberts and Horgan-Kobelski is generated by a computer algorithm that iterates the free critical point using Newton's Method on the

cubic polynomial for  $\lambda \in [-2,2] \times [0,3.5]i$ . This picture is shown in the parameter space, in which the algorithm takes the parameter value and computes the corresponding free critical point and iterates it for a certain user defined amount of iterations. The Mandelbrot-like picture is made of 4 colors; 1 unique color for each root (green for the root at -1 and red for the root at 1), blue for the fixed point (the other root), and black (which is difficult to see in the diagram). If the orbit of the critical point reaches within a certain distance of a certain root it is colored the corresponding color of that root, either green if the orbit is close to -1 or red if the orbit is close to 1. If the orbit of the critical point converges to the fixed point  $\lambda$ , then it is colored blue, the corresponding color of the fixed point. If the orbit does not converge to a root after a given amount,  $\xi$ , of iterations then it is colored black.

It is important to note that when a parameter value is taken from the center or middle area of the triangle-like region, the critical orbit starting with  $\frac{\lambda}{3}$  will converge to the nearest root  $\lambda$ , as depicted by the blue area. As the parameter value moves closer to either the boundary arc or the spine the critical point is now equidistant from more than one root and will be close to forming an isosceles triangle and will have trouble converging. This "trouble" is shown by the beetle-like decorations in the previous figure given by Roberts and Horgan-Kobelski. At the base of each of these "beetles" is a small Mandelbrot set. These Mandelbrot sets are partially the motivation for this paper.



Monochrome Mandelbrot-like picture in the parameter space



The above "zoomed in" pictures of the spine in the parameter space show evidence of baby Mandelbrots Sets.

The coloring in these diagrams is similar to that of the parameter space on page 12. In this case, however, convergence to a specific root is ignored. The parameter value is colored red if the orbit of the critical point converges to a root, yellow if it takes many iterations to converge to a root, or black if the orbit of the critical point does not converge to a root. The shading in the diagram corresponds to the rate of convergence to a root. These diagrams were originally computed by Curry, Garnett, and Sullivan[1983], and Blanchard[1991], but referenced in Roberts and Horgan-Kobelski's[2004] paper.

### 3.3. Restricting $\lambda$ -values to the imaginary axis:

Most of the results in this paper come from restricting the  $\lambda$ -values to the case when  $\lambda = \beta i, \beta \in \mathfrak{R}$ . It is because of the similarity conjugacy that the dynamics along each triangular arc are equivalent. Because the solid spine and each arc are equivalent, it is easiest to study the isosceles case by restricting our studies to the imaginary axis or vertical spine in the parameter space.

With  $\lambda = \beta i$  we have  $p_\lambda(z) = (z^2 - 1)(z - \beta i)$  and  $p'_\lambda(z) = 3z^2 - 1 - 2\beta iz$  with  $z = x + yi$ . It follows that  $p_\lambda(yi)$  is pure imaginary by the following short computation:

$$p_\lambda(yi) = ((yi)^2 - 1)(yi - \beta i) = (-y^2 - 1)(yi - \beta i)$$

This computation shows a real number multiplied by an imaginary number and therefore yielding a strictly imaginary number. Therefore the imaginary axis in the dynamic space is invariant under Newton's Method where  $\lambda = \beta i$ . In other words, every iteration of a pure imaginary number yields pure imaginary output whenever the parameter  $\lambda$  is strictly imaginary. Moreover, since the free critical point  $r = \lambda/3 = \beta i/3$  is also pure imaginary,

we can restrict our study to a rational map of one variable. This map is determined by computing the imaginary part of  $R_\beta(y) \equiv \text{Im}(N_{\beta i}(yi))$  and is given by:

$$R_\beta(y) = \frac{2y^3 - \beta y^2 + \beta}{3y^2 - 2\beta y + 1} \quad (1)$$

where  $\beta$  is our new real valued parameter and  $y$  is a real number corresponding to the imaginary part of the original number  $z$ . The free critical point is now  $r = \beta/3$  and we are interested in the orbit of all initial seeds, including the critical point, under  $R_\beta$  for varying values of  $\beta$ .

$$\text{Notice that } R_\beta(\beta) = \frac{2(\beta)^3 - \beta(\beta)^2 + \beta}{3(\beta)^2 - 2\beta(\beta) + 1} = \frac{2\beta^3 - \beta^3 + \beta}{3\beta^2 - 2\beta^2 + 1} = \frac{\beta^3 + \beta}{\beta^2 + 1} = \frac{\beta}{1} = \beta$$

Therefore  $\beta$  is a fixed point of  $R_\beta(y)$ . This was expected since  $\beta i$  is fixed under the original Newton's Method as one of the roots.

Similarly to  $N'_p(z)$  in section 2, we

$$\text{have } R'_\beta(y) = \frac{2(y - \beta)(3y - \beta)(y^2 + 1)}{(3y^2 - 2\beta y + 1)^2} \Rightarrow R'_\beta(\beta) = 0, R'_\beta(\beta/3) = 0 \quad \text{This confirms that}$$

$y = \beta$  is a super attracting fixed point and  $r = \beta/3$  is the free critical point. Finally

solving  $R_\beta(y) = y$  for the fixed points of  $R_\beta$ , we obtain the cubic

equation  $y^3 - \beta y^2 + y - \beta = (y - \beta)(y^2 + 1) = 0$ , confirming that  $y = \beta$  is the only fixed point.

It is important to note that along each of the arcs and solid spine only 2 colors will be present. On the right arc the colors will be "green" and black; green corresponding to the orbits converging to the root at -1 and black corresponding to the orbits not converging to the root. On the left arc the colors will be "red" and black; red corresponding to the

orbits converging to the root at 1 and black corresponding to the orbits not converging to the root. On the solid spine or imaginary axis the colors will be “blue” and black; blue corresponding to the orbits converging to the fixed point  $\beta$  and black corresponding to the orbits not converging to the fixed point. Given a parameter value  $\lambda_1$  on the solid spine with map  $N_{p_{\lambda_1}}$  then there exists a parameter value  $\lambda_2$  with conjugate map  $N_{p_{\lambda_2}}$  on the left arc and then there exists a parameter value  $\lambda_3$  with conjugate map  $N_{p_{\lambda_3}}$  on the right arc. Examining the dynamics of the critical point  $\lambda_1/3$  under its map  $N_{p_{\lambda_1}}$  is equivalent to examining the dynamics of the critical point  $\lambda_2/3$  under the conjugate map  $N_{p_{\lambda_2}}$  or examining the dynamics of the critical point  $\lambda_3/3$  under the conjugate map  $N_{p_{\lambda_3}}$ . From this we can say that if the orbit  $\lambda_1/3 \rightarrow \lambda_1$ , then the orbit of  $\lambda_2/3 \rightarrow 1$  for  $\lambda_2$  on the left arc, or  $\lambda_3/3 \rightarrow -1$  for  $\lambda_3$  on the right arc. And similarly, if  $\lambda_1$  was colored black, so would  $\lambda_2$  and  $\lambda_3$ . So, if there was a different color other than the black and the corresponding color of the vertex opposite of the base of the isosceles triangle made by the roots, then there must exist a flaw in the algorithm for drawing the picture.

## 4. The Dynamics of $R_\beta$

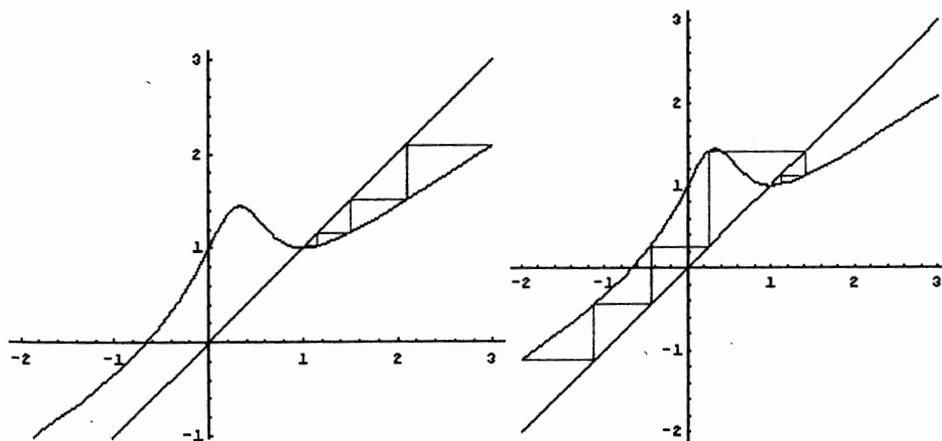
### 4.1 Known Dynamics

The dynamics of all initial  $y$  values can explained for  $0 < \beta < \sqrt{3}$ . However, first  $\beta = 1$  is chosen to illustrate the dynamics of this interval of  $\beta$ .

Graphical iteration is a useful way to visualize the behavior of initial seeds. It is easiest to think of graphical iteration as starting on the reference line  $y_n = y_{n+1}, (y = x)$ .

The starting points on the reference line represent the initial seeds. Each iterate is equivalent to going from the reference line to the graph of  $R_\beta$  and then back to the reference line as depicted by the red “cobweb” designs in the following figures. This graphical “cobweb” technique is widely used in examining dynamics and will be used in this paper to show specific dynamics for different  $\beta$  values.

Any orbit with initial  $x$  value starting to the right of  $y = \beta = 1$ , will necessarily converge monotonically to the fixed point at  $y = 1$ . Similarly, any orbit with initial  $x$  value starting to the left of the fixed point will increase in value until the iterates pass the value  $y = 1$ , which in turn will then decrease monotonically and converge to the fixed point. The only exception is points that are the pre-images of the fixed point at  $\beta = 1$ . These pre-image points will monotonically converge and land exactly on the fixed point after a finite number of iterations.



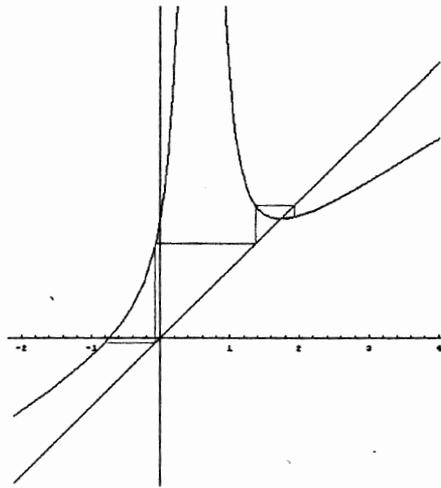
Here are two plots with  $\beta = 1$ , showing 2 different orbits that get attracted to the fixed point. However, these are not orbits of the critical point.

*Claim:* Similarly, for  $0 < \beta < \sqrt{3}$ , we can extend the previous example to show that all orbits converge to the fixed point  $\beta$ , including the free critical point. The point  $y = \beta$  is thus a globally attracting fixed point.

*Explanation:* First notice that the poles (asymptotes) of  $R_\beta, y_\pm = (\beta \pm \sqrt{\beta^2 - 3})/3$  are real only if  $\beta \geq \sqrt{3}$ . At  $\beta = \sqrt{3}$  there is a single pole at the free critical point  $r = \sqrt{3}/3$ . Thus, the denominator of  $R_\beta$  is strictly positive in the regime of  $0 < \beta < \sqrt{3}$ . From this it is straight-forward to compute that  $R_\beta(y) > y$  when  $y < \beta$ , and  $R_\beta(y) < y$  when  $y > \beta$ . Similar to the  $\beta = 1$  case, we know that any point to the right of the fixed point will monotonically converge to the fixed point at  $\beta$ . We also know that all points to the left of the fixed point  $\beta$  (with the exception of the pre-images of the fixed point) will also converge to the fixed point at  $\beta$  by first increasing until the orbit passes the critical point and then monotonically decreasing to the fixed point. The pre-images of the fixed point will converge monotonically to the fixed point as in the specific case for  $\beta = 1$ .

Visually, these maps for  $0 < \beta < \sqrt{3}$  correspond to parameter values in the middle area of the triangle in the parameter space on page 9. Note that the whole triangular area is one color in the parameter space; the color of the fixed point  $\lambda$ .

## 4.2 The Interesting Dynamics Begin



Plot of  $R_\beta$  with  $\beta = \sqrt{3}$

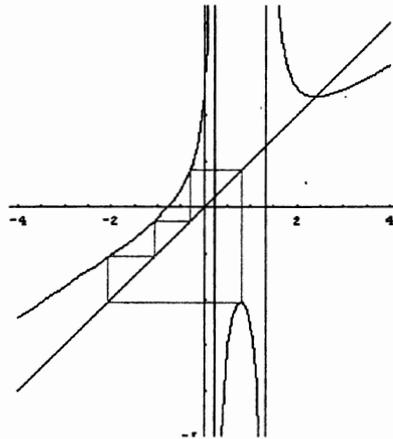
At  $\beta = \sqrt{3}$  a double root develops in the denominator in the graph of  $R_\beta$  in equation (1).

This causes the free critical value to be undefined, so therefore Newton's Method cannot iterate because of an undefined portion of the formula. However, using graphical

iteration it is still obvious that other initial seeds, besides the free critical point, will necessarily converge to the fixed point. All points that are pre-images of the critical point will also be undefined after some finite number of iterations and therefore undefined for Newton's Method.

For the parameter  $\beta > \sqrt{3}$  is where the interesting dynamics occur. A change in dynamics causes the overall question "What happens to all values of  $y$  for all  $\beta$ " to become a more difficult question. This change in dynamics occurs when asymptotes develop in the graph of  $R_\beta$  in equation (1). The asymptotes correspond to the denominator of  $R_\beta$  being able to be factored into two real roots, whereas with  $0 < \beta < \sqrt{3}$ , the denominator has no real roots. These asymptotes allow orbits to have different fates other than only strictly going to the fixed point  $\beta$ . Roberts and Horgan-Kobelski[2004] showed that many parameter values exist for corresponding maps which have superattracting periodic orbits.

#### 4.2.1 A Specific Example



For example, this figure shows the graph of  $R_\beta$  for  $\beta = 2.3963854$  along with a superattracting period 4 orbit of the critical point. This parameter value is chosen to correspond within numerical accuracy to a map in which a superattracting period 4 orbit exists. This particular orbit is also referred to as a superattracting period 4 orbit with rotation number  $\frac{1}{4}$ . A rotation number for a periodic orbit is a rational number  $\frac{p}{q}$  where  $q$  is the primary period of the cycle and  $p$  is the number of points on the orbit that land between the asymptotes of  $R_\beta, y_\pm$ . So, for example, the previous diagram of  $R_\beta$  with  $\beta = 2.39638545$ , is a super attracting period 4 cycle with rotation number  $\frac{1}{4}$  because the orbit lands exactly on the critical point after 4 iterates and only one point on the orbit lies in the middle interval between the asymptotes.

#### 4.2.1.a. Basin of Attraction for the Fixed Point for $\beta = 2.39638545$

For  $\beta > \sqrt{3}$ ,  $R_\beta$  has two unique asymptotes  $y_-$  and  $y_+$ , where  $y_-$  is the left asymptote and  $y_+$  is the right asymptote. The free critical point  $r = \beta/3$  is always between the two asymptotes. It is important to note that, for any  $\beta > \sqrt{3}$ , anytime an orbit lands to the right of the right vertical asymptote  $y_+$ , the orbit will necessarily converge to the fixed point at  $\beta$ . The interval  $(y_+, \infty)$  is called the “immediate basin of attraction” for the fixed point at  $\beta$ . The “immediate basin of attraction for the fixed point” means that every point in this interval will have an orbit that necessarily converges to the fixed point. Further classification of “basin of attraction for the fixed point” refers

to the immediate basin of attraction and pre-images of the immediate basin of attraction for the fixed point.

Basically, there are three different types of orbits; orbits that start in or eventually land in the immediate basin of attraction for the fixed point, orbits that land in the basin of attraction for periodic orbits and pre-periodic orbits, and orbits that are neither and that most likely land on the dividing points between the previous two basins of attraction. Graphically, the immediate basin of attraction for the fixed point is depicted as the gray shaded region in the following diagram.

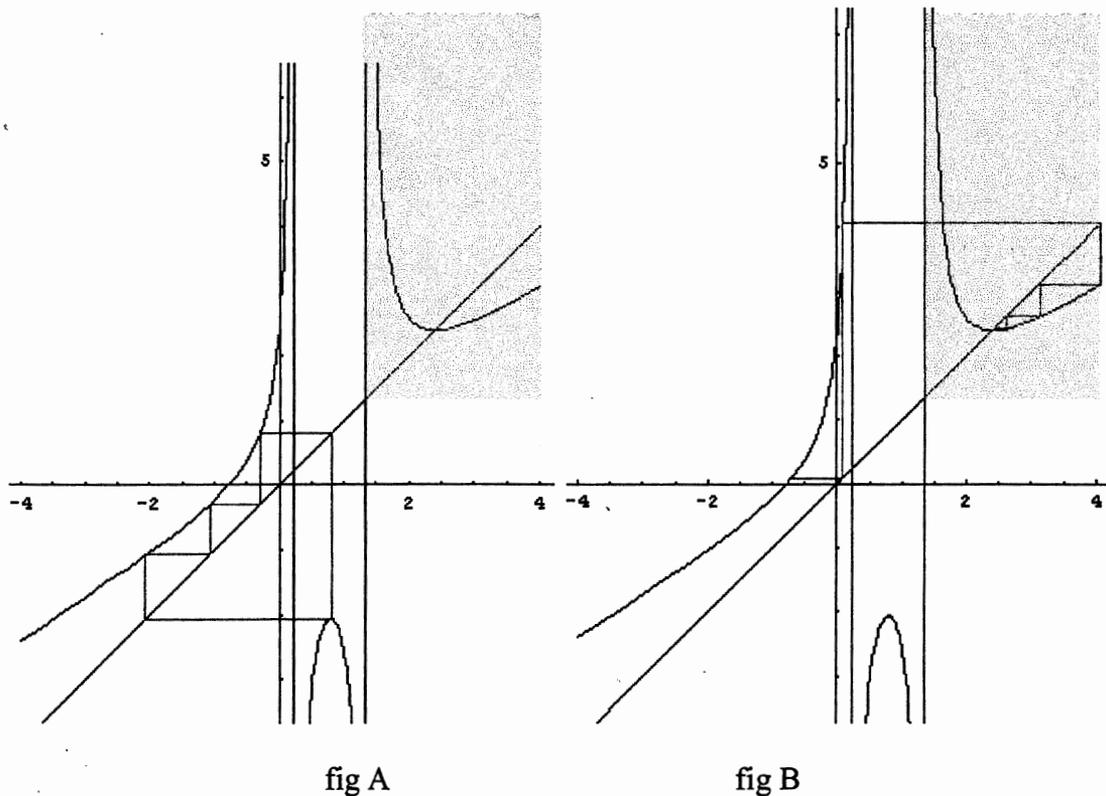
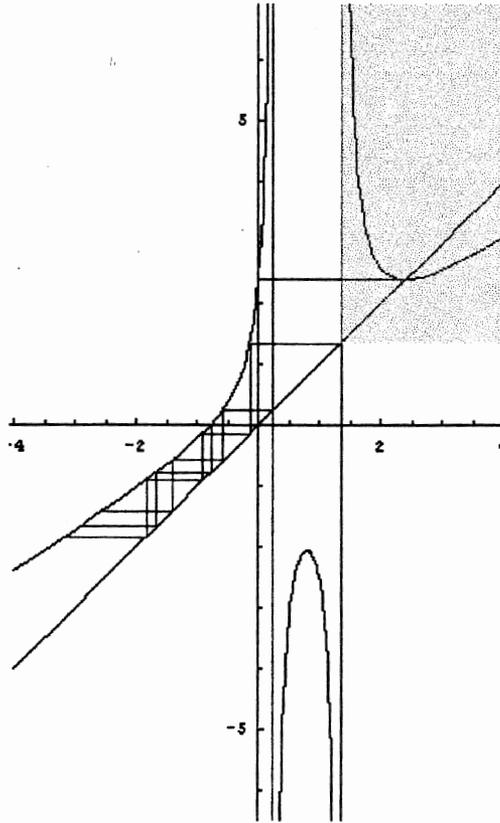


Figure A shows the immediate basin of attraction and the existence of a periodic orbit that doesn't converge to the fixed point.

Figure B shows an orbit that lands in the immediate basin of attraction and therefore necessarily converges to the fixed point.

By taking successive pre-images of the interval  $(y_+, \infty)$ , or equivalently, of the left and right asymptotes, one can find some initial seeds or even regions that will necessarily converge to the fixed point.



Pre-images of the basin of attraction for the fixed point with  $\beta = 2.39638545$

In the previous figure a few pre-images of the basin of attraction are displayed on the graph of  $R_\beta$ . The blue line represents pre-image points of the fixed point. The green and red lines represent the pre-image interval for the basin of attraction where the red is the right endpoint of the pre-image interval, and green is the left endpoint. The blue line lies in the interior of this interval. Successive pre-images past the few shown in the diagram become more difficult to compute. Multiple pre-image intervals now appear as pre-images at each step. For example pre-images can exist in the middle interval  $(y_-, y_+)$  and

therefore successive pre-images after this middle interval will land near the earlier pre-images. This phenomenon will be seen in the following pre-image analyzer plots.

#### 4.2.2 General Case for Varying $\beta$ Values:

##### 4.2.2.a. Basin of Attraction of the Fixed Point for Varying $\beta$ Values:

Computing the pre-images of the basin of attraction for one value of  $\beta$  is not too difficult, but to know what happens to all  $y$ 's and all  $\beta$  values is a much more difficult question to answer. One way to look at many values of  $\beta$  and  $y$  at the same time is the following figure C. Figure C was constructed using a simple computer program called Pre-Image Analyzer. Using this program, the user defines the increment for the  $y$  and  $\beta$  values (both use the same increment), the interval of  $y$  and  $\beta$ , and number of total iterations of  $x$ . The figure was constructed by iterating many  $y$  values for many different  $\beta$  values.

```
Pre-Image Analyzer
Enter starting b value :=1.732051
Enter ending b value :=5
Enter starting y value :=-10
Enter increment of y value:=.0001
Enter number of iterations for each y:=12
```

Figure 1. User Interface for Pre-Image Analyzer

Figure C shows the Pre-Image Analyzer depicting some  $(\beta, y)$  points that converge to the fixed point. The Pre-Image Analyzer takes the initial seeds ( $y$  values) and iterates them

using the  $R_\beta$  function. The points are then colored by the number of iterations it took the orbit of each seed to enter the immediate basin of attraction for the fixed point  $(y_+, \infty)$ .

Each color corresponds to a given amount of iterations, as given by the following table:

Iterates	Color
0	Black
1	Yellow
2	Green
3	Light Blue
4	Dark Blue
5	Purple
6	Pink

Each successive color is a pre-image of the previous color and can be best visualized by the colored near-horizontal bands. The large black area at the top of the figure C corresponds to points that started in the immediate basin of attraction for the fixed point.

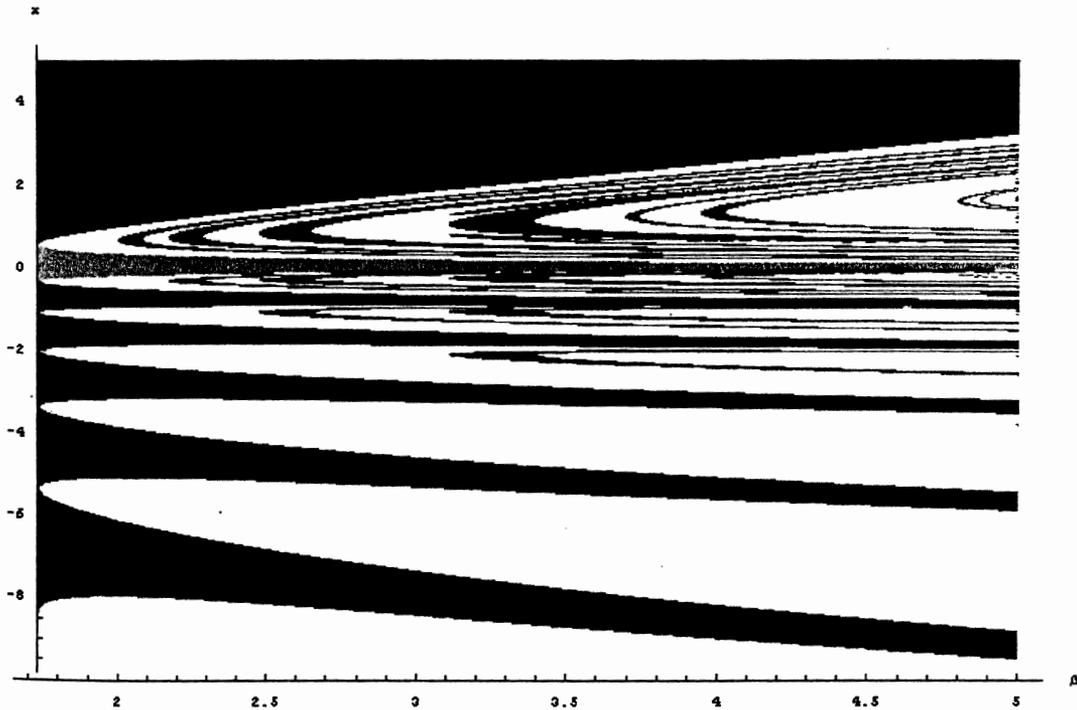


Figure C shows pre-images of the basin of attraction

While this figure shows information for orbits of different  $y$  values that land in the immediate basin of attraction after 6 iterates, it doesn't reveal any information for the  $y$  values that are not colored. Such points could be attracted to a periodic orbit, could be points with iterates that do not enter any basin of attraction, or the points could be in the basin of attraction for the fixed point after more iterations (because 6 iterations is not very many iterations). The parabola-like features in figure C come from the pre-image intervals landing in the middle interval  $(y_-, y_+)$ , as mentioned in the previous section. Note the purple parabola originates around the parameter value  $\beta = 2.2$ . This phenomenon corresponds to the 5<sup>th</sup> pre-image interval going through the middle interval  $(y_-, y_+)$ . More pre-images of the immediate basin of attraction would be nicer. However, it would be very difficult to differentiate the different bands with just a limited amount of colors.

By increasing the number of iterates it takes to get to the interval  $(y_+, \infty)$ , neglecting the colors, and just testing whether or not the initial seed eventually entered the interval  $(y_+, \infty)$ , the following figure D shows a similar, yet different result.

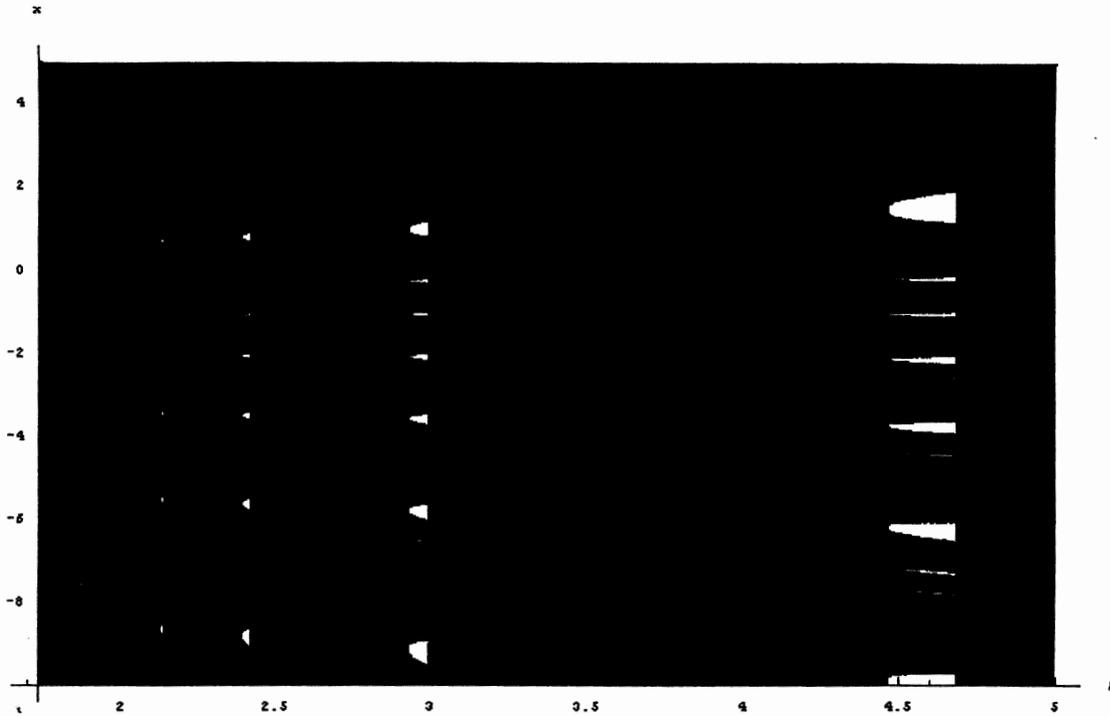


Figure D shows all  $y$  values that entered the basin of attraction after 50 iterates or earlier. Figure D suggests further iterations leave behind white “untouched” areas, or areas of  $y$  values did not converge to the fixed point. This picture suggests that most initial seeds converge to the fixed point, but the white areas imply that this is not the only case and that other dynamics occur.

#### 4.2.2.b. Periodic Orbits of $R_\beta$

It turns out that much of the white in figure D is explained by the existence of attracting periodic orbits. As proved by Roberts and Horgan-Kobelski[2004], there exists a parameter value for which the corresponding map has superattracting periodic cycles of period  $n$  for all  $n$  greater than 2. They stated that there exists a decreasing sequence,  $\xi_n$ , toward  $\sqrt{3}$ , of parameter values that correspond to period- $n$  superattracting orbits. The following is some of their proof included here for completeness.

$$\frac{d}{d\beta}(R_\beta(\beta/3)) = \frac{(\beta^2 + 9)^2}{9(\beta^2 - 3)^2} > 0.$$

Taken with that fact that the  $\lim_{\beta \rightarrow \sqrt{3}^+} R_\beta(\beta/3) = -\infty$ , it follows that the image of the critical point  $r$  is decreasing towards  $-\infty$  as  $\beta$  decreases towards  $\sqrt{3}$ . This is the key fact behind the proposition.

Let  $n$  be some period with  $n \geq 2$ . Consider  $g_n(\beta) = R_\beta^n(\beta/3) - \beta/3$  as a function of the parameter  $\beta$ . A root of  $g_n(\beta)$  gives a period  $n$  cycle for  $R_\beta$  and this cycle is superattracting because it contains the critical point  $r$  in its orbit. We can claim without explanation that there exists a value  $\xi_n$  with  $\sqrt{3} < \xi_n < 3\sqrt{3}$  such that  $g_n(\beta)$  is a continuous function in  $\beta$  on the open interval  $(\sqrt{3}, 3\sqrt{3})$  and that

$$\lim_{\beta \rightarrow \sqrt{3}^+} g_n(\beta) = -\infty \text{ and } \lim_{\beta \rightarrow \xi_n^-} g_n(\beta) = \infty$$

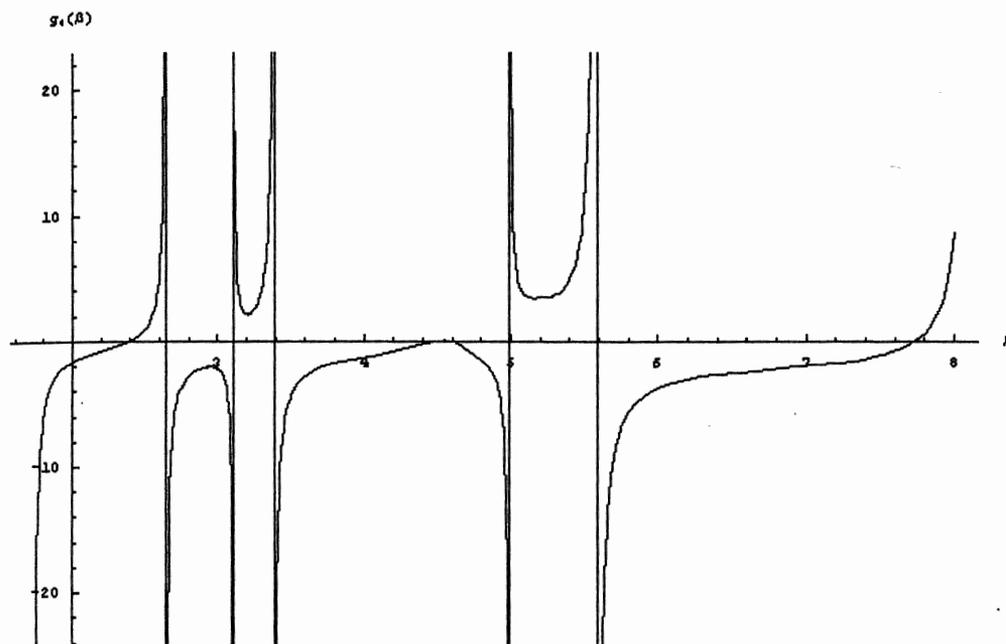
The intermediate value theorem then gives the existence of a root for  $g_n(\beta)$  in the interval  $(\sqrt{3}, \xi_n)$ .

#### 4.2.2.c. Finding Superattracting Periodic Orbits

Finding periodic orbits is equivalent to solving  $g_n(\beta) = R_\beta^n(\beta/3) - \beta/3 = 0$ , for  $\beta$ . That is, given a specific  $n$ , find out for which  $\beta$  values does the  $n$ th iterate of the critical point  $\beta/3$  come back to itself. Because of the complexity of the  $R_\beta$  function, this is most easily done using graphical techniques. For example, looking for period 4 orbits would be equivalent to solving  $g_4(\beta) = R_\beta^4(\beta/3) - \beta/3 = 0$ , or

$R_\beta(R_\beta(R_\beta(R_\beta(\beta/3)))) - \beta/3 = 0$ . Obviously the composition is very complicated.

Looking at the graph of  $g_4(\beta)$ , we can see where it crosses the axis. These crossings are where the superattracting periodic orbits lie. Note there exists a 5<sup>th</sup> root at  $\beta = 24.5$  which is not displayed in the following graph.

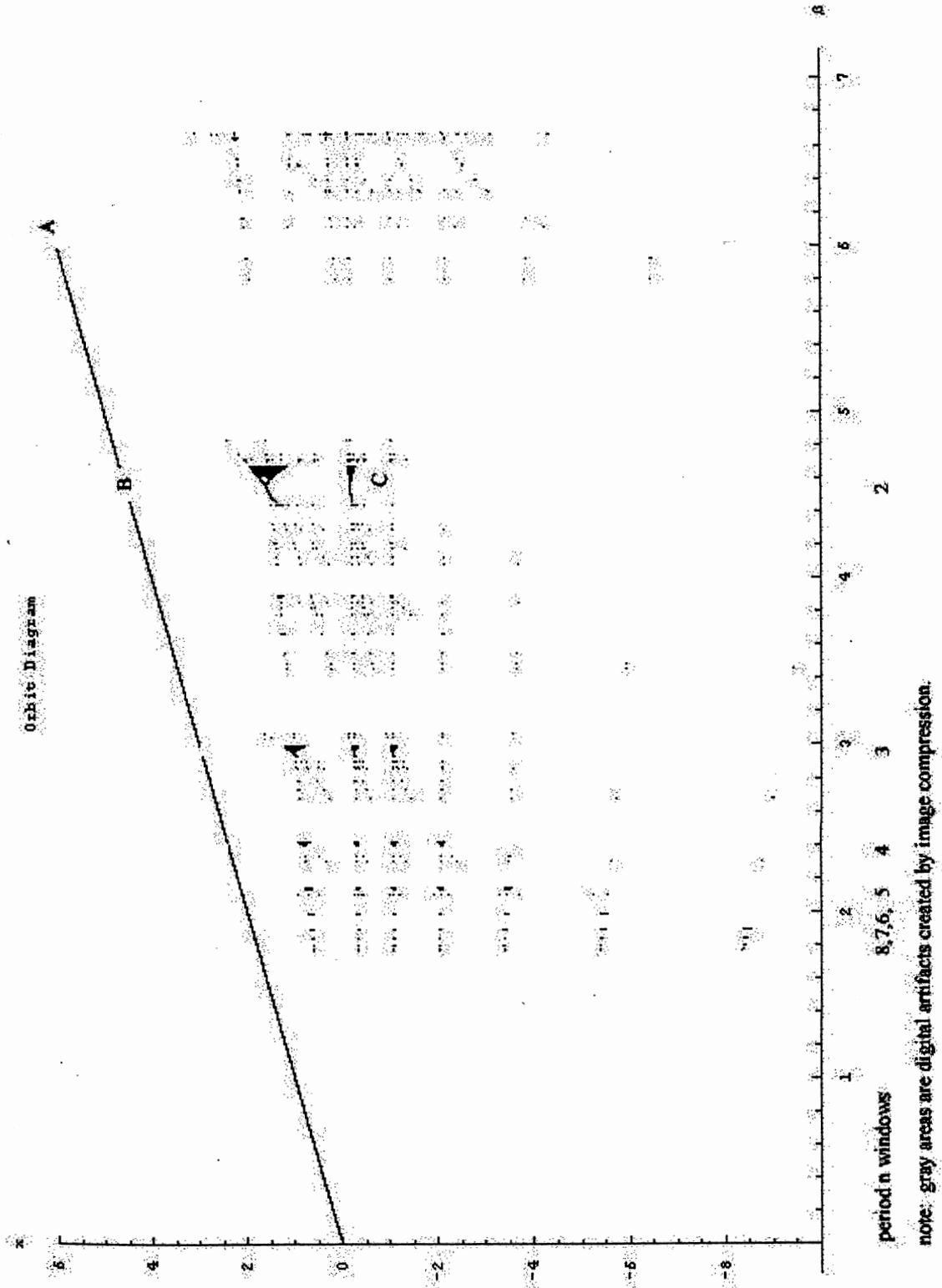


The  $\beta$  values for these periodic orbits are as follows.

$\beta$	2.39638545	4.5001626	4.58784584	7.737193049
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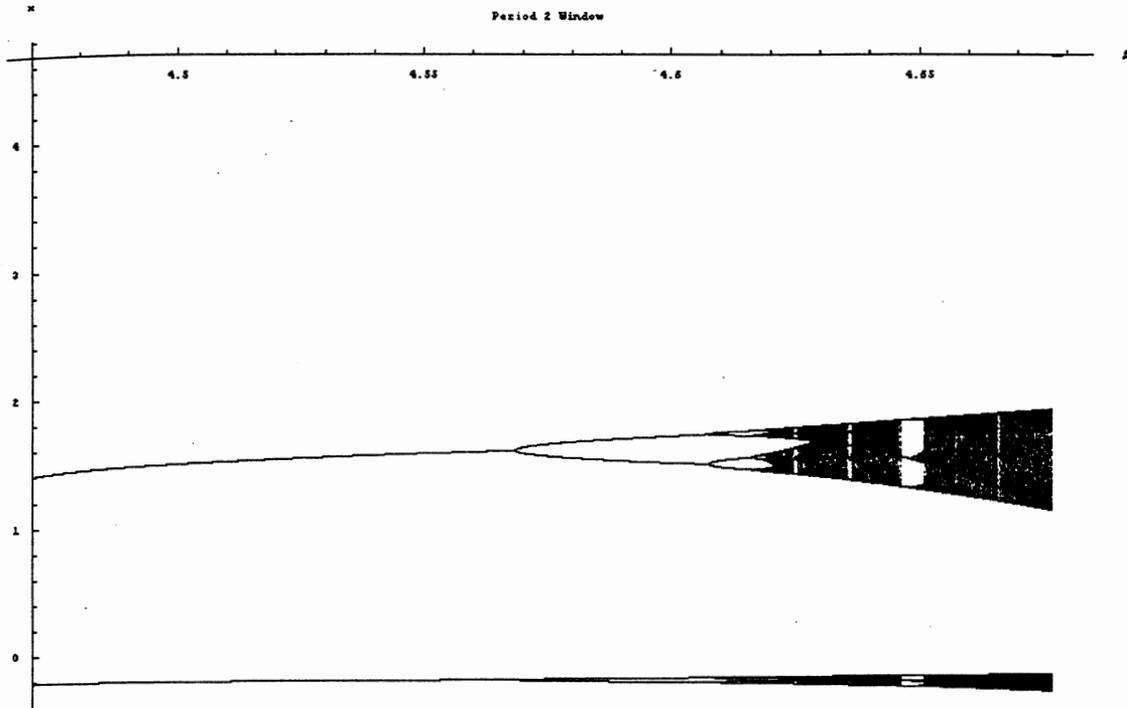
These  $\beta$  values were found using the *FindRoot* command in *Mathematica*, which ironically, uses Newton's method to find the root. Using the *FindRoot* command requires a very accurate initial guess of the root in this application. One should also note that  $\beta=4.5001626$  is a duplicate from the period 2 cycle. In fact, every superattracting orbit with composite period will have a superattracting orbit for any parameter value where its factors have a superattracting periodic orbit. Another example would be that there could be a superattracting period 6 orbit at any  $\beta$  value for which there exists a superattracting period 2 or a superattracting period 3.

### 4.2.2.d. Orbit Diagram



The fate of the critical orbit is most easily shown using an orbit diagram.

This orbit diagram was computed by iterating  $R_\beta(\beta/3)$ , the orbit of the free critical point under  $R_\beta$ .  $R_\beta$  was iterated 1500 times from  $\beta/3$  but only the last 50 points were recorded therefore removing most transient points. The points in the diagram are fates of orbits that have been attracted to periodic cycles, fixed points, or it could be an orbit that never converges to either phenomena. These phenomena also agree with those of the pre-image analyzer previously shown. One major feature in the orbit diagram is the “line” with gaps in it. (A) The solid portions of the line suggest that, after many iterations, the orbit of the critical point  $\beta/3$  converged to the fixed point. For the interval  $0 < \beta \leq \sqrt{3}$ , the only feature is a solid line on the orbit diagram. This was expected because we know the fate of all the orbits in this region is convergence to the fixed point. (B) Gaps in the line can be referred to as periodic windows or areas that the orbit of the critical point did not go to the fixed point after  $n=1500$  iterations. (C) The bifurcation sequence of period doublings in the periodic windows are very similar to the well known bifurcations for the complex quadratic family  $q_c(x) = x^2 + c$ . One difference is that the window “opens and closes” with different behavior. For example, the following period 2 window opens and closes on intervals of orbits going to the fixed point (depicted as a white area). The period 3 window in the  $q_c(x) = x^2 + c$  case “opens and closes” on chaotic behavior (depicted as a blackened out area). Many of these characteristic features can be seen in the following diagram of period 2 window.

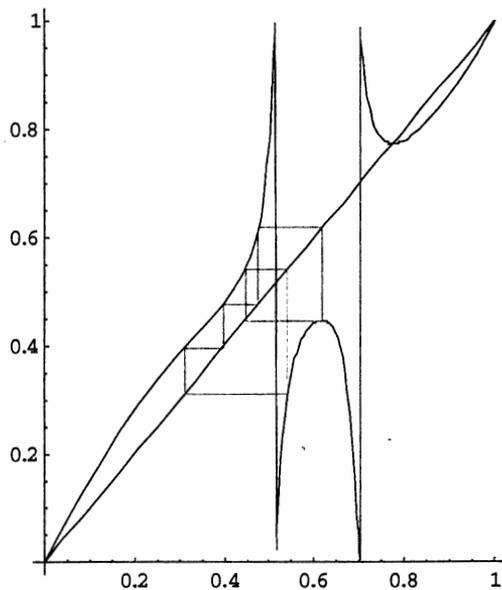


A few parameter values  $\lambda_n = \beta_n i$  which correspond to super attracting periodic orbits are labeled in the lower region of the orbit diagram are given by Roberts and Horgan-Kobelski in the following table:

$n$	$\beta_n$	$n$	$\beta_n$	$n$	$\beta_n$
2	4.50016263	6	1.98209864	10	1.77722708
3	2.93806919	7	1.892459429	20	1.73281876
4	2.39638545	8	1.83640072	25	1.732151907
5	2.13108922	9	1.80052219	50	1.732050812

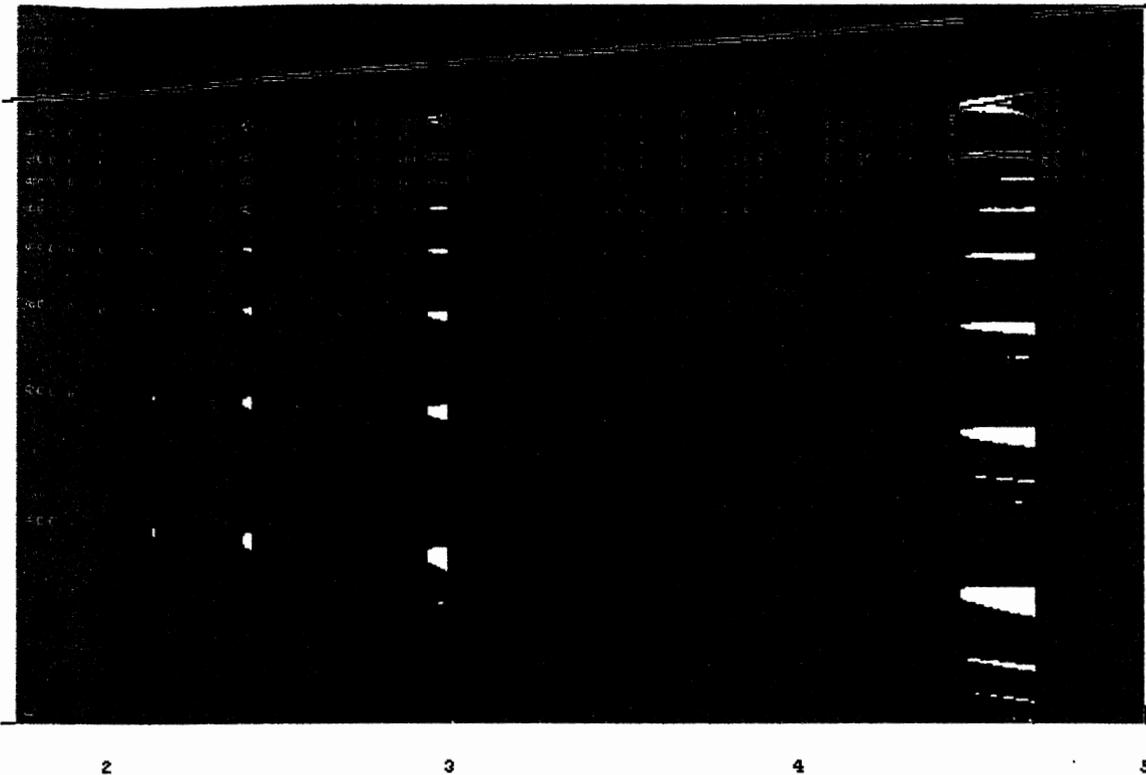
It is easy to see in the orbit diagram that many other interesting orbits exist in just the small interval for  $\beta, \sqrt{3} < \beta < 5$ . For example, there also exists a superattracting period 4 orbit at  $\beta \approx 4.58784584$  (which can be seen in the period 2 window) and a superattracting period 5 orbit at  $\beta \approx 3.88913990787$  and a superattracting period 6 orbit

at  $\beta \approx 3.667823184$ . (see the following figure) The superattracting period 5 and 6 parameter values correspond to places where the orbit of the free critical point lands between the poles  $y_{\pm}$  at least once before returning to the critical point to complete the cycle. In other words, these superattracting periodic orbits have different rotation numbers than the listed superattracting orbits in Roberts and Horgan-Kobelski table (which are all "1/n" superattracting period n orbits). Further classification of superattracting periodic orbits and their corresponding rotation numbers will be examined later in the paper.



Orbit of  $\beta/3$  with  $\beta \approx 3.667823184$ , rotation number  $\frac{2}{6}$

#### 4.2.2.e Superimposed Orbit Diagram and Pre-Image Diagram



#### 4.2.2.e Superimposed Orbit Diagram and Pre-Image of the Fixed Point Diagram

Superimposing the orbit diagram and the pre-image of the fixed point diagram it is apparent that small white areas exist around the orbit diagram. Every  $y$  value in the small white areas will be attracted to the black points lying in the interior of the particular white area in that window. This phenomenon is most easily visualized by a point starting at a low white area in the period 2 window. Successive iterates of this point would eventually bring it into the top two white areas in the window.

## 5. Circle Maps:

Circle maps are a widely used tool for analyzing functions for many reasons. We

chose to use a circle map in this research paper was for the following reasons:

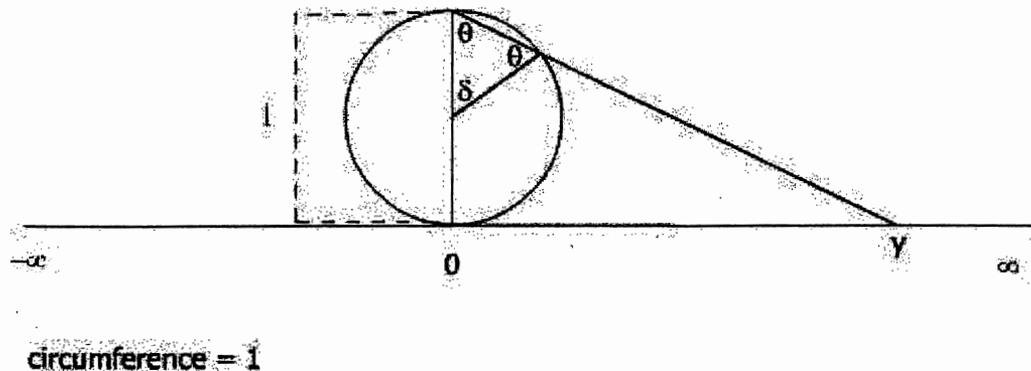
First, by using a circle map all  $y$  values in  $(-\infty, \infty)$  can be examined by compactifying the set to a more useable interval, in our case  $[0,1)$ . This is especially helpful graphically.

Secondly, we can now examine the point at infinity,  $\{\infty\}$ . Using the circle map graphically we can see that the slope through the point at infinity is greater than 1 (lies above the reference line  $y=x$ ) and therefore it is a repelling fixed point (see figure on page 38). This is consistent with known results about Newton's Method "at infinity".

Thirdly, the circle map is a continuous function because infinity is <sup>identified with</sup> ~~mapped to~~ zero.

Fourth, by using a lift of the circle map we can construct a continuous function which allows us to examine rotation numbers of the superattracting periodic orbits.

### 5.1. Defining a Conjugate Map:



By taking any point on the real number line, one can compute the angle  $\theta$  from the circle in the above diagram and therefore find the corresponding fraction of the way around the

circle by dividing the angle  $\theta$  by  $2\pi$ . It is from this transformation where the name circle map originates. By using this transformation we can examine just the interval  $[0,1)$  instead of  $(-\infty, \infty)$  by using a homeomorphism. The function for the transformation and its inverse are as follows:

$$\theta = h(y) = \frac{\tan^{-1}\left(\frac{y}{\pi}\right)}{\pi} + .5$$

$$y = h^{-1}(\theta) = \pi \tan(\pi(\theta - .5))$$

After the transformation we have the following:

$h: (-\infty, \infty) \rightarrow (0,1)$ . Then we extend this to  $\{\infty\} \cup (-\infty, \infty)$  by defining the image of the point at infinity  $\{\infty\} \mapsto \{0\} \equiv 1$ .

Finally, we have  $h: (-\infty, \infty) \cup \{\infty\} \rightarrow S^1$ .

By having a one-to-one, onto, and continuous function with a continuous inverse we can claim that examining orbits in  $[0,1) \in S^1$  is identical to examining numbers or orbits in the interval  $\{\infty\} \cup (-\infty, \infty) \equiv \overline{\mathbb{R}}$ . These two sets are homeomorphic because of the continuous and invertible function  $h$  joining the two sets. The homeomorphism  $h$  allows us to define a circle map  $r_\beta$  that is conjugate to  $R_\beta$ .

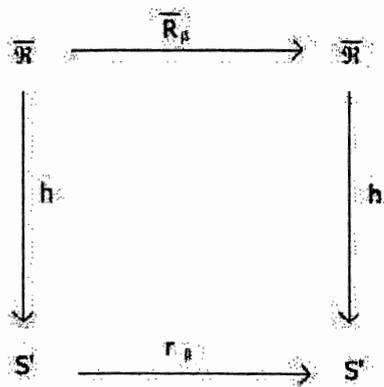
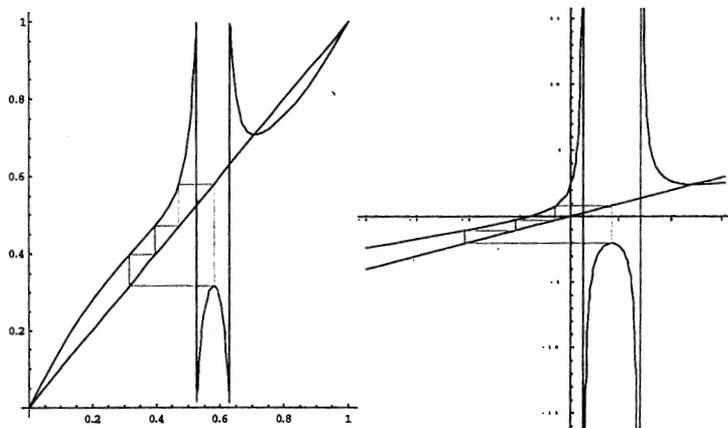


Diagram showing conjugate sets  $R$  and  $S^1$

The function  $r_\beta$  can be defined by going “up, over and down” in the previous diagram.

Analytically this is  $r_\beta = h(R_\beta(h^{-1}(\theta)))$ .  $r_\beta$  is now defined as  $r_\beta : S^1 \rightarrow S^1$ . Since  $r_\beta$  is conjugate to  $R_\beta$  we can examine the dynamics of  $r_\beta$  instead of  $R_\beta$ . This is especially helpful because  $r_\beta$  is now a continuous function on a compact space.

By using this conjugacy we can now look at the same  $R_\beta$  over all  $y$  values.



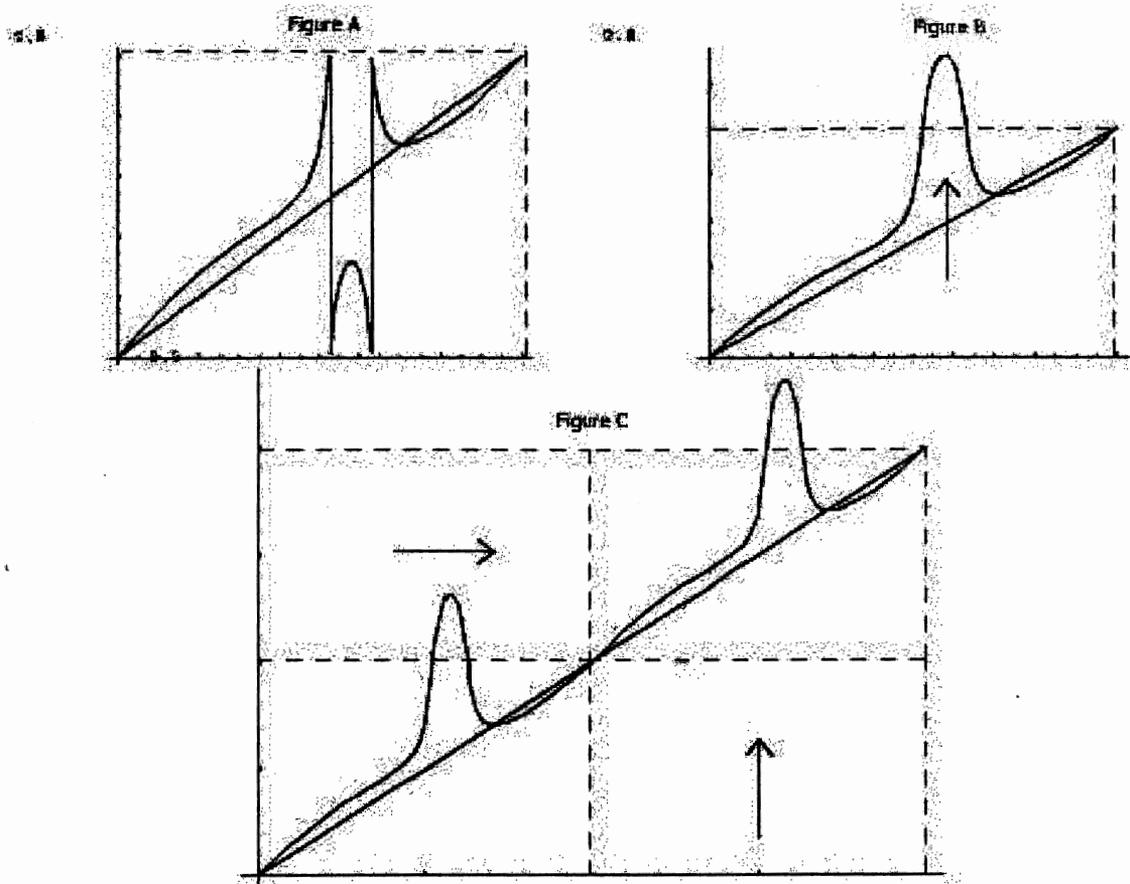
$r_\beta$  versus the  $R_\beta$  function for  $\beta = 2.39638545$

## 5.2. Rotation Numbers

A standard tool for defining rotation numbers on any circle map is its lift. By taking the circle map  $r_\beta$  we can create a continuous function for  $\hat{\theta} \in \mathbb{R}^1$  using a “lift” of our circle map. Let the “lift”  $\hat{r}_\beta$  of  $r_\beta$  be defined as follows:

$$\hat{r}_\beta(\hat{\theta}) = \begin{cases} r_\beta(\hat{\theta}) + \text{floor}(\hat{\theta}) & \text{floor}(\hat{\theta}) < \hat{\theta} \leq h(y_-) \\ r_\beta(\hat{\theta}) + \text{floor}(\hat{\theta}) + 1, h(y_-) + \text{floor}(\hat{\theta}) < \hat{\theta} < h(y_+) + \text{floor}(\hat{\theta}) \\ r_\beta(\hat{\theta}) + \text{floor}(\hat{\theta}) & h(y_+) + \text{floor}(\hat{\theta}) \leq \hat{\theta} < 1 + \text{floor}(\hat{\theta}) \end{cases}$$

Where  $h(y_-)$  and  $h(y_+)$  ( $y_\pm$ ) are the asymptotes of the function  $r_\beta$  and the  $\text{floor}(\hat{\theta})$  function truncates any non-integer part of  $\hat{\theta}$ . The new variable  $\hat{\theta}$  is defined for all real numbers and this allows us to plot the function outside of the interval  $[0,1)$ . The following functions were plotted using *Mathematica*.



A lift is constructed by taking the original circle map on  $[0,1)$  fig(A) and pushing the middle up (fig B) and then make copies of it and shift it up and over, preserving the continuity of the graph and function (fig C)(on the interval  $[0,2]$ ). This shifting is represented in the function as the  $\text{Floor}[\hat{\theta}]$  part of  $\hat{r}_\rho(\hat{\theta})$ . The floor function takes the point where the orbit landed and adds to it the corresponding vertical shift. The integers (corresponding to  $y = \infty$ ) are also a fixed points, which can be seen by the graph (fig. C) where the graph crosses the line  $y = x$ . These fixed points at infinity ~~is~~ are repelling because the slope of the line at that point is greater than 1, causing all iterates to repel away from the point. Iterates that leave the interval  $[0,1)$  (or the dotted boxes) are outside

of the interval  $[0,1) \cong S^1$  and are considered to be going around the circle more than once.

Periodic orbits that go around the circle more than once before returning to the initial

point have rotation numbers  $\frac{m}{n}$  where  $m > 1$  is the number of times the orbit has gone

“around” and  $n$ , the period, is the number of iterates before returning to the initial point.

Formally, Rotation numbers are defined as the following:

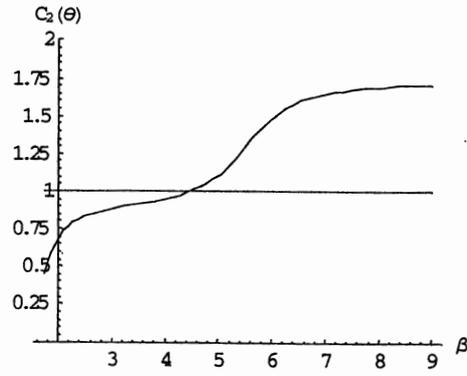
$$\rho(\theta, \beta) = \lim_{n \rightarrow \infty} \frac{\widehat{r}_\beta^n(\widehat{\theta}) - \widehat{\theta}}{n}$$

In words, this is the average number of times around the circle each iterate travels.

Note that  $\widehat{\theta} \in [0,1]$ , but  $\widehat{r}_\beta^n(\widehat{\theta})$  can be greater than 1. This can be seen in figure C.

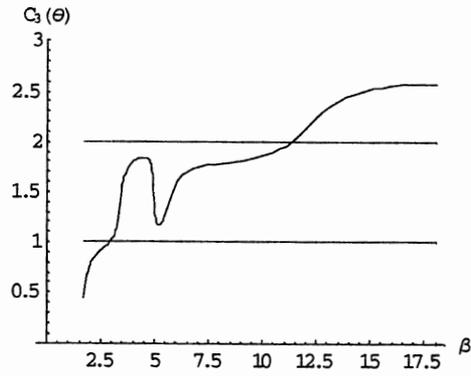
### 5.3. Revisiting Superattracting Periodic Orbits

An alternate approach to finding the superattracting periodic orbits is to use the lift of the circle map instead of  $g_n(\beta) = R_\beta^n(\beta/3) - \beta/3$ . Using a similar technique used on  $g_n(\beta)$ , we graph  $c_n(\beta) = \widehat{R}_\beta^n(h(\beta/3)) - h(\beta/3)$  and look for the roots of the equation. However, in this case we do not look for intersections along the x-axis but along the lines  $y=1, y=2, y=3, \dots, y=n-1$ . These lines correspond to the number of rotations around the circle it takes for the orbit to return to the critical point. This graph allows us to not only locate super attracting periodic orbits like in the previous graph, but also provides information about the number of rotations the orbit takes to complete the cycle.



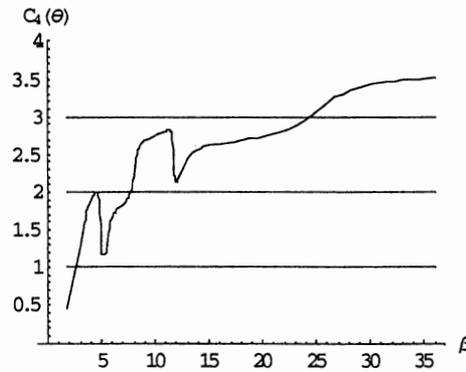
rotation number	1/2
count	1

Period 2,  $c_2(\beta)$ , 1 crossing.



rotation number	1/3	2/3
count	1	1

Period 3,  $c_3(\beta)$ , 2 crossings



rotation number	1/4	2/4	3/4
count	1	3	1

Period 4,  $c_4(\beta)$ , 5 crossings

From just the period 2, period 3, and period 4 graphs of  $c_n(\beta)$  it is obvious that other superattracting orbits exist other than just the  $\frac{1}{n}$  orbits (which are the first orbits on the left side of the graph). For example, we can see that for period 4 orbits there exist superattracting orbits with rotation numbers  $\frac{1}{4}, \frac{3}{4}$  at  $\beta = 2.39638545, \beta = 24.5$ , respectively. However, according to the graph of  $c_4(\beta)$ , there are 3 superattracting periodic orbits with rotation number  $\frac{2}{4}$  at

$\beta = 4.50016263, 4.58784584, \text{ and } 7.737193049$ .

Note the two crossings where the graph appears to be tangent to the line  $y=2$ . Repeating this procedure for other periods, one can count the number of crossings and the  $\beta$  value for each crossing. See the table in the appendix for  $\beta$  values corresponding superattracting orbits and their rotation numbers up to period 7.

## 6. Conjectures from graphical evidence:

Graphical evidence provides some insight into the rotation numbers of periodic orbits.

The following 5 conjectures follow from this evidence:

**Conjecture 1:** The  $\frac{1}{n}$  orbits are unique.

This is shown as only 1 crossing of the graph of  $c_n(\beta)$  over the line  $y=1$  for each fixed  $n$ .

It should be noted that the existence of these  $\frac{1}{n}$  orbits has been proven by Roberts and Horgan-Kobelski.

**Conjecture 2:** The  $\frac{n-1}{n}$  orbits exist and are unique.

It is also apparent that the graph of  $c_n(\beta)$  always crosses the line  $n-1$  for each fixed  $n$ .

This is shown in the previous graphs for  $n=2,3,4$ . If a large enough  $\beta$  value is chosen, then the graph of  $c_n(\beta)$  appears to cross the line  $y=n-1$ .

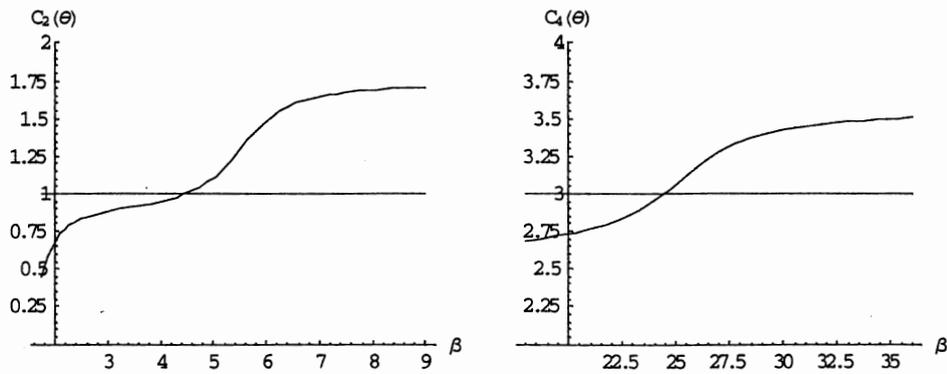
**Conjecture 3:** The  $\frac{n-1}{n}$  superattracting period- $n$  orbit exists in

$$\beta \in [4.50016263 \cdot 2^{n-2}, 4.50016263 \cdot 2^{n-1}].$$

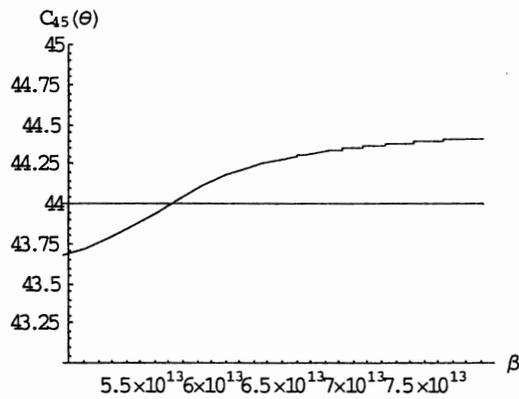
An estimated interval for the crossing of  $c_n(\beta)$  and the line  $y = n-1$  is

$$[4.50016263 \cdot 2^{n-2}, 4.50016263 \cdot 2^{n-1}], \text{ with } n \text{ being the desired period.}$$

The following figures support this conjecture.



$c_2(\beta)$  and  $c_4(\beta)$  plotted on the interval  $[4.50016263 \cdot 2^{n-2}, 4.50016263 \cdot 2^{n-1}]$



$c_{45}(\beta)$  plotted on the interval  $[4.50016263 \cdot 2^{n-2}, 4.50016263 \cdot 2^{n-1}]$

The interval  $[4.50016263 \cdot 2^{n-2}, 4.50016263 \cdot 2^{n-1}]$  was estimated from the distance from the original period 2 orbit at  $\beta = 4.50016263$  and the first few  $\frac{n-1}{n}$  superattracting orbits.

**Conjecture 4:**

Given any rotation number  $\frac{m}{n}$ , there exists a parameter value  $\beta$ , for which there exists a rotation number  $\frac{m}{n}, n > m$ .

This conjecture follows from the existence of the  $\frac{1}{n}$  orbits and the conjectured existence of the  $\frac{n-1}{n}$  orbits. It would then follow from the Intermediate Value Theorem that the graph of  $c_n(\beta)$  must cross every line between  $y=1$  and  $y=n-1$  because the function is continuous. Therefore, since the graph of  $c_n(\beta)$  crossed all lines from 1 to  $n-1$  for all  $n$ , we have all rotation numbers in  $(0,1)$ .

**Conjecture 5: Superattracting Period-n Orbits Triangle**

This conjecture is the strongest conjecture and is based from conjectures 1 and 2.

Conjecture 5 is similar to conjecture 4 but is much stronger. Conjecture 5 hints at the existence of all rotation numbers along with symmetry of periodic orbits and their rotation numbers.

The following diagram is based on a counting of each of the observed super attracting period-n orbits. The left-most number on the triangle is the number of orbits with

rotation number  $\frac{1}{n}$ , and the right-most number is the number of orbits with rotation

number  $\frac{n-1}{n}$ . The values in between are the  $\frac{2}{n}, \frac{3}{n}, \dots, \frac{n-2}{n}$  rotation number orbits.

period n												Superattracting Orbits					
2																	1
3																	2
4																	5
5																	8
6																	19
7																	34
8																	73
9																	140
10																	287

The values in the triangle can be computed using the following recursive formula:

$$a(k, n) = a(k-1, n-1) + a(k, n-1) + (-1)^n$$

where

$$a(n-1, n) = 1, (\text{conjecture} - 2)$$

$$a(1, n) = 1, (\text{conjecture} - 1)$$

$$n = 1, 2, 3, \dots$$

$$k = 2, 3, \dots, n-2$$

Here n is the period (row), k is the number of rotations (column) in the triangle.

The values and rotation numbers have been counted and verified up to period 9. The numbers for the period 10 row are conjectured values using the above formula.

## 7. Future Studies

A few areas for future studies:

### 1. Programming:

- Design a program that shows pre-images iterates of the basin of the attraction for the fixed point for a specific b value. Incorporate a slider for the b values to visualize the dynamics.

- Rewrite the orbit diagram program so that any function can be input by the user using a graphical user interface. Design the program so that the user has “instant” graphics along with zooming features
- Write a program that automates the counting of the roots using the circle map and specific rotation numbers.

## 2. Mathematics:

- Prove that  $1/n$  and  $n-1/n$  superattracting periodic orbits are unique.
- Prove that all rotation numbers in the interval  $[0,1)$  exist using a circle map
- Use the Farey Fraction argument and symbolic coding for proving all rotation numbers up to  $1/2$ . Then extend to  $[0,1)$ .
- Describe mathematically the dynamics between the pre-image colored bands in the pre-image analyzer diagram and the superattracting periodic bands.

## References:

Graphical Analysis.m (mathematica package for cobweb iteration)

Thomas LoFaro

See the following website:

<http://library.wolfram.co.jp/infocenter/MathSource/387/>

“Newton’s Versus Halley’s Method: An Approach Via Complex Dynamics”

by Dr. Gareth E Roberts and Jeremy Horgan-Kobelski, *International Journal of Bifurcation and Chaos*, 2004.

“A First Course in Dynamical Systems”, Devaney, Robert L. , Westview Press Perseus Books Publishing, 1992.

“The Dynamics of Newton’s Method”, Blanchard, Paul.

*Proceedings of Symposia in Applied Mathematics*, Volume 49, 1994.

J.H Curry, L. Garnett and D. Sullivan, On the iteration of a rational function: Computer experiments with Newton’s Method, *Communications in Mathematical Physics*, 91(1983), pp 139-154.

Orbit Diagram Program written in C.

```
#include <stdio.h>
#include <math.h>

#define Func(y) (2*y*y*y-b*y*y+b)/(3*y*y-2*b*y+1)

float y,x1,b1,b2,cp,step;
int points, eventual;
void main()
{
FILE *fpTemps;

fpTemps=fopen("morepoints.txt","w");//open file to write points in to be later read by mathematica

printf("Orbit Analyzer\n\n");
printf("Enter starting b value :==");
scanf("%f",&b1);
printf("Enter finishing b value :==");
scanf("%f",&b2);
printf("Enter increment of b value:==");
scanf("%f",&step);
printf("Enter number total # points:==");
scanf("%d",&points);
printf("Enter number of eventual points:==");
scanf("%d",&eventual);
        fprintf(fpTemps, "{{%f,%f}",b1,b1);                //starts file in correct format w/ critical pt

        //printf("the free critical value is:==%f\n",b/3);

while(b1<b2)
{
        cp=b1/3;                //initializes critical point to b/3
        y=cp;
        // fprintf(fpTemps,"%f",b);                //writes the b val
        // fprintf(fpTemps,"%f",cp);                //writes the critical point into file

        for(int i=0;i<points;i++)
        {
                x1=(2*y*y*y-b1*y*y+b1)/(3*y*y-2*b1*y+1);

                if(i>(points-eventual)){                //Only
print the eventual behavior
                //printf("the %d iteration is value is:==%f\n",i,x1); //screen output
                fprintf(fpTemps, "{{%f,%f}",b1,x1);
                }
                y=x1;

        }

        // fprintf(fpTemps, "\n");
        b1=b1+step;                //increment step
}

fprintf(fpTemps, " ");
fclose(fpTemps);//Close file
}
```

pedrag vubaz jxjrnpeck. nC, EL@CFOUT 732800

7ndik 8xbl.0  
7ndik 8xbl.0

7Ae3i 7rpyo 7\*7j\*7b\*7y7(737)\*72\*7y7f

7atx7p7d7j7l727c7s7e7

7atx7p7j7l7

7r7x7d7c7s7e7a7a7x7d7g

7d7i7a7r7

{

7I7I7f7k7p7s

7c7o7d7l7:7c7o7d7e7o7u7s7r7t7o7b7e7s7u7r7v7a7l7o7r7a7h7a7n7t7a7  
7c7o7d7l7:7c7o7d7e7o7u7s7r7t7o7b7e7s7u7r7v7a7l7o7r7a7h7a7n7t7a7  
7c7o7d7l7:7c7o7d7e7o7u7s7r7t7o7b7e7s7u7r7v7a7l7o7r7a7h7a7n7t7a7  
7c7o7d7l7:7c7o7d7e7o7u7s7r7t7o7b7e7s7u7r7v7a7l7o7r7a7h7a7n7t7a7  
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7c7o7d7l7:7c7o7d7e7o7u7s7r7t7o7b7e7s7u7r7v7a7l7o7r7a7h7a7n7t7a7

7S7a7h7a7:7f7e7i7h7e7o7u7r7e7o7r7a7t7&amp7r7o7s7i7n7a7n7t7o7p7e7d7i7n7t7o7  
7S7a7h7a7:7f7e7i7h7e7o7u7r7e7o7r7a7t7&amp7r7o7s7i7n7a7n7t7o7p7e7d7i7n7t7o7  
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7S7a7h7a7:7f7e7i7h7e7o7u7r7e7o7r7a7t7&amp7r7o7s7i7n7a7n7t7o7p7e7d7i7n7t7o7  
7S7a7h7a7:7f7e7i7h7e7o7u7r7e7o7r7a7t7&amp7r7o7s7i7n7a7n7t7o7p7e7d7i7n7t7o7  
7S7a7h7a7:7f7e7i7h7e7o7u7r7e7o7r7a7t7&amp7r7o7s7i7n7a7n7t7o7p7e7d7i7n7t7o7  
7S7a7h7a7:7f7e7i7h7e7o7u7r7e7o7r7a7t7&amp7r7o7s7i7n7a7n7t7o7p7e7d7i7n7t7o7  
7S7a7h7a7:7f7e7i7h7e7o7u7r7e7o7r7a7t7&amp7r7o7s7i7n7a7n7t7o7p7e7d7i7n7t7o7  
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7S7a7h7a7:7f7e7i7h7e7o7u7r7e7o7r7a7t7&amp7r7o7s7i7n7a7n7t7o7p7e7d7i7n7t7o7

7n7i7t7(7l7o7s7t7u7g7e7&amp7n7d7x7d7i7n7t7)  
7n7i7t7(7l7r7a7s7t7a7d7i7g7o7v7d7o7==7)  
7k7a7n7(797d7727d7)  
7n7i7t7(7l7r7a7s7t7a7d7i7g7o7v7d7o7==7)  
7k7a7n7(797d7727d727)  
7n7i7t7(7l7r7a7s7t7a7d7i7g7o7v7d7o7==7)  
7k7a7n7(797d7727d707)  
7s7u7p7(7E7n7t7b7i7d7i7g7o7v7d7o7==7)  
7c7a7n7(797d7727d727)  
7n7i7t7(7l7r7a7s7t7a7d7i7g7o7v7d7o7==7)  
7k7a7n7(797d7727d707)  
7n7i7t7(7l7r7a7s7t7a7d7i7g7o7v7d7o7==7)  
7k7a7n7(797d7727d707)

7v7a7l7(7r727)

7i7=7x7;  
7i7=(7b7-s7q7t747\*7y7b7-27)7;  
7i7=(7b7-k7r7(7b7-77)7;  
7y7=7a73  
7c7o7

7i7n7i7d7z7e7o7r7e7d7o7p7o7:7o7t7

7f7(7r7o7b777y7n7g7"7e7k7v7a7e7b7e7(7s7i7n7h7f7i7s7a7n7d7b7e7x7==7)7&amp7n7t7e7l7p7o7r7t7(7==7)7f7e7r7g7i7s7a7n7y7k7e7x7==7%7f7r7"7b7a7c7o7;  
7i7=7b7;  
7d7o7w7v7a7t7a7l7u7p7o7r7s7x7i7l7(7r7i7s7t7)7h7e7n7g7i7c7f7e7r7g7i7s7a7n7y7k7e7x7

7y7o7y7g7o7o7h7e7f7o7  
7h7g7=7)

7v7a7l7(7r7x7)

Pre-Image Analyzer program written in C, BLACKOUT VERSION

```

#include <stdio.h>
#include <math.h>

#define Func(y) (2*y*y*y-b*y*y+b)/(3*y*y-2*b*y+1)

double y,x1,x2,x,b1,b2,cp,step;
double b,a,c,xin;
int i,points, eventual,go,flag;
void main()
{
FILE *fpTemps;
FILE *fpTemps1;
FILE *fpTemps2;
FILE *fpTemps3;
FILE *fpTemps4;
FILE *fpTemps5;
FILE *fpTemps6;
FILE *fpTemps7;
FILE *fpTemps8;
FILE *fpTemps9;

fpTemps=fopen("symbolics.txt","w");
fpTemps1=fopen("a.txt","w"); //open file to write points in to be later read by mathematica
fpTemps2=fopen("b.txt","w"); //open file to write points in to be later read by mathematica
fpTemps3=fopen("c.txt","w"); //open file to write points in to be later read by mathematica
fpTemps4=fopen("d.txt","w"); //open file to write points in to be later read by mathematica
fpTemps5=fopen("e.txt","w"); //open file to write points in to be later read by mathematica
fpTemps6=fopen("f.txt","w"); //open file to write points in to be later read by mathematica
fpTemps7=fopen("g.txt","w"); //open file to write points in to be later read by mathematica
fpTemps8=fopen("h.txt","w"); //open file to write points in to be later read by mathematica
fpTemps9=fopen("i.txt","w"); //open file to write points in to be later read by mathematica

fprintf(fpTemps1,"{}"); //Starts the file in the correct format for mathematica to read in a list plot
fprintf(fpTemps2,"{}"); //Starts the file in the correct format for mathematica to read in a list plot
fprintf(fpTemps3,"{}"); //Starts the file in the correct format for mathematica to read in a list plot
fprintf(fpTemps4,"{}"); //Starts the file in the correct format for mathematica to read in a list plot
fprintf(fpTemps5,"{}"); //Starts the file in the correct format for mathematica to read in a list plot
fprintf(fpTemps6,"{}"); //Starts the file in the correct format for mathematica to read in a list plot
fprintf(fpTemps7,"{}"); //Starts the file in the correct format for mathematica to read in a list plot
fprintf(fpTemps8,"{}"); //Starts the file in the correct format for mathematica to read in a list plot
fprintf(fpTemps9,"{}"); //Starts the file in the correct format for mathematica to read in a list plot

printf("Pre-Image Analyzer\n\n");
printf("Enter starting b value :=");
scanf("%lf",&b);
printf("Enter ending b value :=");
scanf("%lf",&b2);
printf("Enter starting x value :=");
scanf("%lf",&xin);
//printf("Enter finishing x value :=");
//scanf("%lf",&x2);
printf("Enter increment of x value :=");
scanf("%lf",&step);
printf("Enter number of iterations for each x:=");
scanf("%d",&go);

while(b<b2)
{
x1=xin;
a=(2*b-sqrt(4*b*b-12))/6; //left asymptote
c=(2*b+sqrt(4*b*b-12))/6; //right asymptote
cp=b/3; //initializes critical point to b/3
y=cp;
//S fprintf(fpTemps,"beta value b:=%lf\nleft asymptote x:=%lf\nright asymptote x:=%lf\n",b,a,cp,c);
x2=b2; //know what happens to all orbits to the right of the right asymptote
c; they go to the f.p.
flag=0;
while(x1<x2)

```

```

{
    x=x1;
for(i=0;i<go;i++)
{
    y=x;
    if(x<=a){ // Start Symbolic Section not used in this paper.
//S      //counter=counter+1;
        fprintf(fpTemps,"O");
    }
    if((x>a)&&(x<cp)){
//S      //counter=counter+1;
        fprintf(fpTemps,"L");
    }
    if(x==cp){
//S      //counter=counter+1;
        fprintf(fpTemps,"C");
    }
    if((x>cp)&&(x<=c)){
//S      //counter=counter+1;
        fprintf(fpTemps,"R");
    } // End Symbolic Section
//S    if(x>c){
        fprintf(fpTemps,"*");

        if(i==51){ //write point if in interval after iteration i= 1
            fprintf(fpTemps1,"%1f,%1f",b,x1);
            flag=1;
        }

        if((i==52)&&(flag!=1)){ //write point if in interval after iteration i= 2
            fprintf(fpTemps2,"%1f,%1f",b,x1);
            flag=1;
        }

        if((i==53)&&(flag!=1)){ //write point if in interval after iteration i= 3
            fprintf(fpTemps3,"%1f,%1f",b,x1);
            flag=1;
        }

        if((i==54)&&(flag!=1)){ //write point if in interval after iteration i= 4
            fprintf(fpTemps4,"%1f,%1f",b,x1);
            flag=1;
        }

        if((i==55)&&(flag!=1)){ //write point if in interval after iteration i= 5
            fprintf(fpTemps5,"%1f,%1f",b,x1);
            flag=1;
        }

        if((i==56)&&(flag!=1)){ //write point if in interval after iteration i= 6
            fprintf(fpTemps6,"%1f,%1f",b,x1);
            flag=1;
        }

        if((i==57)&&(flag!=1)){ //write point if in interval after iteration i= 7
            fprintf(fpTemps7,"%1f,%1f",b,x1);
            flag=1;
        }

        if((i==58)&&(flag!=1)){ //write point if in interval after iteration i= 8
            fprintf(fpTemps8,"%1f,%1f",b,x1);
            flag=1;
        }

        if((i==59)&&(flag!=1)){ //write point if in interval after iteration i= 9
            fprintf(fpTemps9,"%1f,%1f",b,x1);
        }
    }
}

```

```

        flag=1;
    }

} //as soon as the function iterates to the right of the right asymptote, y+, it goes to the FP

x=(2*y*y*y-b*y*y+b)/(3*y*y-2*b*y+1);
//rot=(double)counter/(double)go; //Symbolic part not used in color diagram.

//end of for-loop
flag=0;
//S    fprintf(fpTemps, "\n",b);

x1=x1+step;

//end of while loop
b=b+step;
//end of b loop

fprintf(fpTemps1, "%1f,%1f}", b,x1);//More formatting for mathematica listplot.
fprintf(fpTemps2, "%1f,%1f}", b,x1);//More formatting for mathematica listplot.
fprintf(fpTemps3, "%1f,%1f}", b,x1);//More formatting for mathematica listplot.
fprintf(fpTemps4, "%1f,%1f}", b,x1);//More formatting for mathematica listplot.
fprintf(fpTemps5, "%1f,%1f}", b,x1);//More formatting for mathematica listplot.
fprintf(fpTemps6, "%1f,%1f}", b,x1);//More formatting for mathematica listplot.
fprintf(fpTemps7, "%1f,%1f}", b,x1);//More formatting for mathematica listplot.
fprintf(fpTemps8, "%1f,%1f}", b,x1);//More formatting for mathematica listplot.
fprintf(fpTemps9, "%1f,%1f}", b,x1);//More formatting for mathematica listplot.

//fprintf(fpTemps, "}");
fclose(fpTemps);//Close file
fclose(fpTemps1);//Close file
fclose(fpTemps2);//Close file
fclose(fpTemps3);//Close file
fclose(fpTemps4);//Close file
fclose(fpTemps5);//Close file
fclose(fpTemps6);//Close file
fclose(fpTemps7);//Close file
fclose(fpTemps8);//Close file
fclose(fpTemps9);//Close file
}

```

Pre-Image Analyzer Program written in C COLOR VERSION

```

#include <stdio.h>
#include <math.h>

#define Func(y) (2*y*y*y-b*y*y+b)/(3*y*y-2*b*y+1)

double y,x1,x2,x,x,b1,b2,cp,step;
double b,a,c,xin;
int i,points, eventual,go,flag;
void main()
{
FILE *fpTemps;
FILE *fpTemps1;
FILE *fpTemps2;
FILE *fpTemps3;
FILE *fpTemps4;
FILE *fpTemps5;
FILE *fpTemps6;
FILE *fpTemps7;
FILE *fpTemps8;
FILE *fpTemps9;

fpTemps=fopen("symbolics.txt","w");
fpTemps1=fopen("a.txt","w"); //open file to write points in to be later read by mathematica
fpTemps2=fopen("b.txt","w"); //open file to write points in to be later read by mathematica
fpTemps3=fopen("c.txt","w"); //open file to write points in to be later read by mathematica
fpTemps4=fopen("d.txt","w"); //open file to write points in to be later read by mathematica
fpTemps5=fopen("e.txt","w"); //open file to write points in to be later read by mathematica
fpTemps6=fopen("f.txt","w"); //open file to write points in to be later read by mathematica
fpTemps7=fopen("g.txt","w"); //open file to write points in to be later read by mathematica
fpTemps8=fopen("h.txt","w"); //open file to write points in to be later read by mathematica
fpTemps9=fopen("i.txt","w"); //open file to write points in to be later read by mathematica

fprintf(fpTemps1,"{"); //Starts the file in the correct format for mathematica to read in a list plot
fprintf(fpTemps2,"{"); //Starts the file in the correct format for mathematica to read in a list plot
fprintf(fpTemps3,"{"); //Starts the file in the correct format for mathematica to read in a list plot
fprintf(fpTemps4,"{"); //Starts the file in the correct format for mathematica to read in a list plot
fprintf(fpTemps5,"{"); //Starts the file in the correct format for mathematica to read in a list plot
fprintf(fpTemps6,"{"); //Starts the file in the correct format for mathematica to read in a list plot
fprintf(fpTemps7,"{"); //Starts the file in the correct format for mathematica to read in a list plot
fprintf(fpTemps8,"{"); //Starts the file in the correct format for mathematica to read in a list plot
fprintf(fpTemps9,"{"); //Starts the file in the correct format for mathematica to read in a list plot

printf("Pre-Image Analyzer\n\n");
printf("Enter starting b value :=");
scanf("%lf",&b);
printf("Enter ending b value :=");
scanf("%lf",&b2);
printf("Enter starting x value :=");
scanf("%lf",&xin);
//printf("Enter finishing x value :=");
//scanf("%lf",&x2);
printf("Enter increment of x value:=");
scanf("%lf",&step);
printf("Enter number of iterations for each x:=");
scanf("%d",&go);

while(b<b2)
{
x1=xin;
a=(2*b-sqrt(4*b*b-12))/6; //left asymptote
c=(2*b+sqrt(4*b*b-12))/6; //right asymptote
cp=b/3; //initializes critical point to b/3
y=cp;
//S fprintf(fpTemps,"beta value b:= %lf\nleft asymptote x:= %lf\nright asymptote x:= %lf\n",b,a,cp,c);
x2=b2; //know what happens to all orbits to the right of the right asymptote
c; they go to the f.p.
flag=0;
while(x1<x2)

```

```

{
    x=x1;
for(i=0;i<go;i++)
{
    y=x;
    if(x<=a){ // Start Symbolic Section not used in this paper.
//S
        //counter=counter+1;
        fprintf(fpTemps,"O");
    }
    if((x>a)&&(x<cp)){
//S
        //counter=counter+1;
        fprintf(fpTemps,"L");
    }

    if(x==cp){
//S
        //counter=counter+1;
        fprintf(fpTemps,"C");
    }

    if((x>cp)&&(x<=c)){
//S
        //counter=counter+1;
        fprintf(fpTemps,"R");
    }
//S
    if(x>c){
        fprintf(fpTemps,"*");

        if(i==0){ //write point if in interval after iteration i= 1
            fprintf(fpTemps1,"%1f,%1f","b,x1);
            flag=1;
        }

        if((i==1)&&(flag!=1)){ //write point if in interval after iteration i= 2
            fprintf(fpTemps2,"%1f,%1f","b,x1);
            flag=1;
        }

        if((i==2)&&(flag!=1)){ //write point if in interval after iteration i= 3
            fprintf(fpTemps3,"%1f,%1f","b,x1);
            flag=1;
        }

        if((i==3)&&(flag!=1)){ //write point if in interval after iteration i= 4
            fprintf(fpTemps4,"%1f,%1f","b,x1);
            flag=1;
        }

        if((i==4)&&(flag!=1)){ //write point if in interval after iteration i= 5
            fprintf(fpTemps5,"%1f,%1f","b,x1);
            flag=1;
        }

        if((i==5)&&(flag!=1)){ //write point if in interval after iteration i= 6
            fprintf(fpTemps6,"%1f,%1f","b,x1);
            flag=1;
        }

        if((i==6)&&(flag!=1)){ //write point if in interval after iteration i= 7
            fprintf(fpTemps7,"%1f,%1f","b,x1);
            flag=1;
        }

        if((i==7)&&(flag!=1)){ //write point if in interval after iteration i= 8
            fprintf(fpTemps8,"%1f,%1f","b,x1);
            flag=1;
        }

        if((i==8)&&(flag!=1)){ //write point if in interval after iteration i= 9
            fprintf(fpTemps9,"%1f,%1f","b,x1);

```

```

        flag=1;
    }

} //as soon as the function iterates to the right of the right asymptote, y+, it goes to the FP

x=(2*y*y*y-b*y*y+b)/(3*y*y-2*b*y+1);
//rot=(double)counter/(double)go; //Symbolic part not used in color diagram.

} //end of for-loop
flag=0;
//S fprintf(fpTemps,"n",b);

x1=x1+step;

} //end of while loop
b=b+step;
} //end of b loop

fprintf(fpTemps1,"%1f,%1f}",b,x1); //More formatting for mathematica listplot.
fprintf(fpTemps2,"%1f,%1f}",b,x1); //More formatting for mathematica listplot.
fprintf(fpTemps3,"%1f,%1f}",b,x1); //More formatting for mathematica listplot.
fprintf(fpTemps4,"%1f,%1f}",b,x1); //More formatting for mathematica listplot.
fprintf(fpTemps5,"%1f,%1f}",b,x1); //More formatting for mathematica listplot.
fprintf(fpTemps6,"%1f,%1f}",b,x1); //More formatting for mathematica listplot.
fprintf(fpTemps7,"%1f,%1f}",b,x1); //More formatting for mathematica listplot.
fprintf(fpTemps8,"%1f,%1f}",b,x1); //More formatting for mathematica listplot.
fprintf(fpTemps9,"%1f,%1f}",b,x1); //More formatting for mathematica listplot.

//fprintf(fpTemps,"");
fclose(fpTemps); //Close file
fclose(fpTemps1); //Close file
fclose(fpTemps2); //Close file
fclose(fpTemps3); //Close file
fclose(fpTemps4); //Close file
fclose(fpTemps5); //Close file
fclose(fpTemps6); //Close file
fclose(fpTemps7); //Close file
fclose(fpTemps8); //Close file
fclose(fpTemps9); //Close file
}

```